
Calibrating the Core Collapse of Self-Interacting Dark Matter Halos with N -body Simulations

Bachelorarbeit

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Abstract

In the framework of self-interacting dark matter (SIDM), the core of a SIDM halo undergoes gravothermal collapse. A computationally cheap way of describing the core collapse is via gravothermal fluid model. The downside of this model is, that it needs to be calibrated against the more accurate, but more expensive, N -body simulations. The goal of this thesis is to calibrate the gravothermal fluid model against N -body simulations for four halos and to construct a universal calibration. The method applied here is to compare the number of enclosed numerical particles from an N -body simulation with the number of expected numerical particles at a given radius in the gravothermal fluid model. It is found, that the halo specific calibration generally works better than the universal one. Although the universal calibration, which is quasi-independent of halo specific properties, is usable, if a halo specific calibration is not known.

Zusammenfassung

Im Rahmen der selbst-wechselwirkenden dunklen Materie (SIDM), erfährt der Kern eines SIDM Halos den gravothermalen Kollaps. Ein rechnerisch günstiger Weg den Kernkollaps zu beschreiben ist mit dem gravothermalen Fluid Modell. Der Nachteil dieses Modells ist, dass dieses gegen die genaueren, aber rechenintensiven, N -Körper-Simulationen kalibriert werden muss. Das Ziel dieser Arbeit ist das gravothermale Fluid Modell gegen N -Körper-Simulationen für vier Halos zu kalibrieren und eine universale Kalibration zu konstruieren. In der eingesetzten Methode wird die Anzahl der eingeschlossenen numerischen Teilchen einer N -Körper-Simulation mit der Anzahl der erwarteten numerischen Teilchen im gravothermalen Fluid Modell bei einem gewissen Radius verglichen. Es ist gefunden worden, dass die Halo spezifischen Kalibrationen im allgemeinen besser sind, als die universelle Kalibration. Obwohl die universale Kalibration, welche quasi-unabhängig von den spezifischen Halo Eigenschaften ist, auch benutzbar ist, falls keine Halo spezifische Kalibration bekannt ist.

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List of abbreviations

CDM $\longleftrightarrow$ cold dark matter

CE $\longleftrightarrow$ Chapman-Enskog theory

CMB $\longleftrightarrow$ Cosmic Microwave Background

DNS $\longleftrightarrow$ dynamic nested sampling

LMFP $\longleftrightarrow$ long mean free path

LSS $\longleftrightarrow$ large-scale structure

RBF $\longleftrightarrow$ radial basis function

SIDM $\longleftrightarrow$ self-interacting dark matter

SMFP $\longleftrightarrow$ short mean free path

1 Introduction

The cosmological standard model predicts the existence of cold dark matter (CDM)[4]. The characterizing property of CDM is that it does not interact at all, neither with baryonic particles nor with itself[12]. While CDM works well on large scales, it has multiple issues on smaller scales[10], [48], [20], [5]. One way to fix these issues at small scales, while reproducing the success of CDM at large scales, is to allow for elastic self-interactions between dark matter particles[41]. A natural consequence of the self-interactions is the gravothermal collapse of the cores of SIDM halos, which results in the formation of a black hole[41]. This mechanism might also resolve the existence of supermassive black holes at high redshift[34], [2]. There are several ways to properly model the gravothermal collapse. A computationally cheap way is the gravothermal fluid model, while a computationally more expensive but more exact method are N -body simulations[3], [45]. In the gravothermal fluid model the dark matter is treated as a virialized fluid and thus the evolution of the fluid can be described by four equations[25]. One of these equations is the Fourier law, which relates the luminosity to the temperature gradient, via the heat conductivity[32]. The form of the heat conductivity is only known in limiting cases and thus needs to be interpolated[33]. First a form of interpolation with only one free parameter was proposed[3]. However it was found, that this is too simple to properly model the gravothermal collapse[29], [45]. So a generalized interpolation was proposed, in which two free parameters α and C are introduced[29]. The evolution of the dark matter halo strongly depends on the choice of these parameters. This means that the fluid model needs to be calibrated against N -body simulations[26], [21], i.e. the free parameters need to be determined by comparing the evolution of SIDM halos in the gravothermal fluid model with the evolution from N -body simulations, thus making the gravothermal fluid model at best comparable in accuracy to the N -body simulations.

The goal of this thesis is to calibrate the gravothermal model for four different SIDM halos. Furthermore, a calibration, which should fit a wide range of halos is constructed and discussed.

The structure of this thesis is outlined in the following. Section 2 is about the physical background of this thesis and begins with a motivation and introduction of CDM. Afterwards, problems of CDM at small scales are explained, which lead to an introduction to SIDM. After that the gravothermal collapse is explained and the four equations governing the gravothermal fluid model are derived. Section 3 is about the numerical methods used in this thesis. It begins with an explanation of the numerical implementation of the gravothermal fluid model and the N -body simulations. The section concludes with an introduction to Bayesian inference and dynamic nested sampling (DNS). Section 4 begins with the implementation of the generalized heat conductivity model into the code used to conduct the gravothermal fluid simulations. Afterwards the explicit simulation setup is presented and the section closes with a detailed description of the method used for the calibration. Section 5 presents the fitting results and discusses them. Finally, section 6 summarizes the results and discussion points of this thesis and gives an outlook on the next steps.

2 Physical basics

2.1 Dark matter

2.1.1 Cosmological context

In the cosmological standard model¹, the energy and matter of the universe can be decomposed into four components: dark energy, dark matter, baryonic matter and radiation, guiding the evolution of the universe. Dark matter is the second most dominant component. This is quantified by the density parameter Ω , which is defined as the ratio of the energy density of a component and the critical density of the universe[4]. In the cosmological standard model it is assumed that the universe is flat, meaning that the total energy density is equal to the critical density[4]. Since energy densities of different components are additive, the respective density parameters all sum up to one. In this case Ω directly describes the percentage of the energy density of a component compared to the total energy density. The density parameter of dark matter is $\Omega_{\text{DM}} = 0.264$ ² [43], compared to the density parameter for baryonic matter $\Omega_{\text{b}} = 0.049$ [43], one can conclude that about 84 % of the matter content of the universe is made up of dark matter. From this fact one can infer that dark matter plays an important role in the evolution of structure of the universe and is hence observable on a wide range of length scales, from dwarf galaxies, to the size of the observable universe, i.e. large-scale structure (LSS) or the Cosmic Microwave Background (CMB)[12].

2.1.2 Cold dark matter

In a CDM model, dark matter behaves like a non-relativistic fluid. In cosmic evolution, this implies that a dark matter fluid has the same equation of state as a (non-relativistic) baryonic fluid, i.e. the dark matter fluid is pressureless or equivalently the equation of state parameter $w = 0$. The property that distinguishes CDM from baryonic matter is that interactions between CDM particles and standard model particles or other CDM particles are negligible, except for gravitational interactions[12].

The structure of CDM halos can be probed with N -body simulations. These simulations help determine a density profile for the halos. A profile, which fits a wide range of halos is the so-called NFW-profile[28], [44]

$$\rho(r) = \frac{\rho_s}{\left(\frac{r}{r_s}\right) \left(1 + \frac{r}{r_s}\right)^2}, \quad (1)$$

where r_s is the scale radius and ρ_s the scale density. Both the scale radius and scale density are halo specific properties, which determine the profile fully. The scale density and radius can be expressed as a function of the halo mass M and the concentration c , defined as[30]

$$c := \frac{r_{200}}{r_s}, \quad (2)$$

where r_{200} is the radius, where the halo density is equal to 200 times the critical density

¹Also called Λ CDM model.

²The dark matter is assumed to be cold, see section 2.1.2.

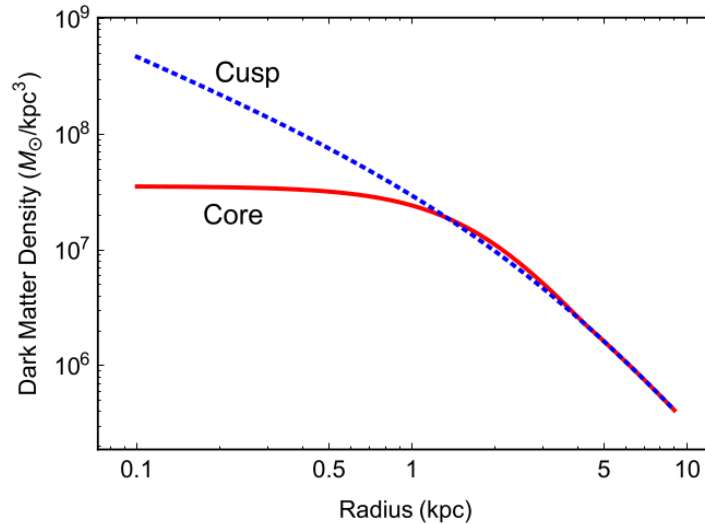


Figure 1: The dark matter density is shown as a function of radius for both a cored and cusp profile. Reprinted from[44].

of the universe. So alternatively a complete description of the halo profile can be given through concentration and mass.

While CDM makes robust predictions on cosmological scales, i.e. the LSS or CMB, it suffers from problems on smaller scales. One of the most prominent ones being the *core-cusp problem*. As can be seen from equation (1), simulations of CDM predict that the logarithmic slope of the density at small radii is $\ln(\rho)/\ln(r) \approx -1$. The profile is said to be *cuspy*[44]. However this contradicts the observation of many low-mass, dark matter dominated, galaxies[10]. These observations imply that $\ln(\rho)/\ln(r) \approx 0$. So a nearly flat density profile, also called *cored*, is observed[44]. An illustration of the problem can be found in figure 1.

A similar, though not identical issue, is the mass-deficit problem. This problem states that the total observed dark matter mass in the center of galaxies is less than what is theoretically predicted. Formally the core-cusp problem and the mass-deficit problem are distinct, since the first one is about the derivative of the density, while the latter is about the integral of the density. Despite that, some authors still point to a common cause for both problems[10]. A review of further prominent problems of CDM can be found in [44], [10].

A natural question that arises is how to make dark matter compatible with observations on small scales. So a number of solutions were proposed, from the inclusion of baryonic feedback to alternative dark matter theories[48]. One of these theories is SIDM, which was investigated in this thesis.

2.1.3 Self-interacting dark matter

The main difference between SIDM and CDM is that collisions between dark matter particles are in a SIDM theory no longer negligible[41]. More specifically, dark matter particles are allowed to scatter elastically[44]. The motivation is that through this interaction, SIDM can reproduce the success of CDM on cosmological scales, since the scattering rate is proportional to the density[1] and the dark matter density is negligible

on cosmological scales[33]. On the other hand SIDM, will lead to drastically different evolution on galactic scales, possibly correcting the problems of CDM[41].

One deviation from CDM is the formation of an isothermal core. Due to the interactions between dark matter particles, heat is initially transferred inwards from the hotter outer parts to the inner cooler part, resulting in the inner region, also called *core*, becoming isothermal. Along with the thermalization of the core, the heating also results in a reduced density, therefore resolving the core-cusp problem[44]. The inward flow of heat is only a temporary process at the beginning of the halo evolution³ and is reversed once the core actually becomes isothermal[29]. This reversed process is called *gravothermal collapse* and explained in section 2.2.

Another deviation from CDM is that, while CDM halos are triaxial, SIDM halos are nearly spherical. The reason for the sphericity is that the trajectories of the dark matter particles are, due to self-interactions, nearly isotropic. Deviations from sphericity are due to rotation[41], the influence of baryons or remaining velocity dispersion anisotropy[1].

2.2 Gravothermal collapse

As described in the previous section, the reduction of the core's density, due to the thermalization, is only a temporary phase. Once the core reaches a maximum size, the heat flows outwards, from the core into the extended halo, which starts the gravothermal collapse. The collapse mechanism described in the following does not only apply to SIDM halos, but more generally to any system, consisting of gravitating point particles, which can interact via elastic collisions. Historically it was first considered in the case of globular clusters. In that case stars were modelled as point particles. However stellar interactions are not fully elastic, so the collapse is eventually halted by the formation of binary star systems. This is however not the case for SIDM halos, since the interactions are purely elastic, as stated in section 2.1.3. This implies that the collapse continues until the formation of a black hole. Hence the gravothermal collapse is also sometimes referred to as *gravothermal catastrophe*. Despite the exact mechanism governing the collapse being rather involved, one can intuitively understand the process by acknowledging that a gravitationally bound system has a negative heat capacity. In the case of an SIDM halo, both the core and the extended halo have a negative heat capacity. Due to the heat transfer from the core into the extended halo⁴, the core will shrink and increase its temperature, while simultaneously cooling the extended halo. The now increased temperature difference will increase the heat flux from the core to the extended halo, triggering a runaway process.

The gravothermal collapse mechanism can also be used to tackle open problems in cosmology and astrophysics; one of these is the existence of supermassive black holes at high redshift. When using the cosmological standard model to explain the existence of black holes with masses of $\mathcal{O}(10^9 M_\odot)$ at redshifts $z \approx 6$, one quickly runs into the problem of needing to make optimistic assumptions about various physical processes like star formation, cooling and black hole spin, as well as accreting closely to the Eddington limit or alternatively having periods of super-Eddington accretion. This problem can potentially be resolved by the gravothermal collapse, since a collapsed SIDM halo in the early universe is a natural candidate for a seed of a supermassive black hole[34].

³For a detailed explanation of the evolution of a SIDM halo, see[29].

⁴After the core became isothermal.

2.2.1 Gravothermal fluid model

In this section the equations governing the gravothermal collapse, in the *gravothermal fluid model*, are derived. The model makes the assumptions that the dark matter can be treated as a virialized fluid, furthermore the interactions between dark matter particles are modelled as isotropic hard-sphere scattering and the halo is assumed to be spherically symmetric. Then the evolution of the fluid can be described by four equations that follow from mass conservation, hydrostatic equilibrium, Fourier's law, and the laws of thermodynamics. The rest of section 2 will derive these four equations. The derivation largely follows the one given by Pollack in his senior thesis[33].

2.2.2 Mass conservation

Under the assumption of spherical symmetry, total mass $M(r)$ enclosed in a radius r is given by:

$$M(r) = 4\pi \int_0^r dr' r'^2 \rho(r'), \quad (3)$$

where $\rho(r)$ is the density. The differential form of equation (3) can be obtained by differentiating the equation with respect to r . This is also the first evolution equation,

$$\boxed{\frac{\partial M}{\partial r} = 4\pi r^2 \rho}. \quad (4)$$

2.2.3 Hydrostatic equilibrium

To derive an equation from the assumption of hydrostatic equilibrium, one starts with the Poisson equation for gravity

$$\Delta\phi = 4\pi G\rho, \quad (5)$$

where ϕ is the gravitational potential and G the gravitational constant. After inserting equation (4) and simplifying the laplacian with the assumption of spherical symmetry, one gets

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \phi}{\partial r} \right) = \frac{G \frac{\partial M}{\partial r}}{r^2}, \quad (6)$$

multiplying by r^2 and integrating yields

$$\frac{\partial \phi}{\partial r} = \frac{GM}{r^2}. \quad (7)$$

Consider a volume V , the total force $\mathbf{F}$ acting on this volume is given by[24]

$$\mathbf{F} = - \int_{\partial V} d^2\mathbf{x} p(\mathbf{x}). \quad (8)$$

Using Gauss integral theorem, one can rewrite equation 8 into an integral over the entire volume[24]

$$\mathbf{F} = - \int_{\partial V} d^2\mathbf{x} p(\mathbf{x}) = - \int_V d^3x \nabla p(\mathbf{x}), \quad (9)$$

where p is the pressure. Hence $\nabla p(\mathbf{x})$ can be interpreted as the force per unit volume. Using that Newtons second law per unit volume is just the density times the acceleration and accounting for gravity with the $-\rho\nabla\phi$ term[24], the equations of motion become

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla p - \rho\nabla\phi. \quad (10)$$

Next one can use the definition of the total time derivative

$$\frac{d}{dt} := \frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla) \quad (11)$$

to write the total time derivative of the velocity vector $\mathbf{v}$ as

$$\frac{d\mathbf{v}}{dt} = \frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v}. \quad (12)$$

Equation (12) can be inserted into (10), to obtain⁵

$$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho}\nabla p - \nabla\phi. \quad (13)$$

Under the assumption of hydrostatic equilibrium $\mathbf{v} = 0$. Then equation (13) becomes

$$\nabla p = -\rho\nabla\phi. \quad (14)$$

Under the assumption of spherical symmetry and after substituting equation (7) for $\partial\phi/\partial r$, one obtains

$$\frac{\partial p}{\partial r} = -\frac{GM\rho}{r^2}. \quad (15)$$

Finally p is expressed through ρ and the (one-dimensional) velocity dispersion ν . The velocity dispersion of a gas is defined as

$$\nu^2 := \frac{k_B T}{m}, \quad (16)$$

where k_B is the Boltzmann constant, T the temperature of the gas and m is the mass of an individual gas particle. Using the ideal gas law in the form

⁵Equation (13) is also known as the Euler equation[24].

$$p = \frac{k_B T \rho}{m}, \quad (17)$$

the pressure and density can be related, via ν , after substituting equation (16) into equation (17),

$$p = \nu^2 \rho. \quad (18)$$

Inserting equation (18) in (15) results in the second evolution equation

$$\boxed{\frac{\partial (\nu^2 \rho)}{\partial r} = -\frac{GM\rho}{r^2}}. \quad (19)$$

2.2.4 Fourier law

The Fourier law relates the heat flux with the temperature gradient[32], under the assumption of spherical symmetry, it is expressed as

$$q_r = -\kappa \frac{\partial T}{\partial r}, \quad (20)$$

where q_r is the radial heat flow and κ is the heat conductivity. The heat flux can be expressed with the luminosity L radiated through a sphere of radius r using the following equation:

$$L := 4\pi r^2 q_r. \quad (21)$$

One can substitute equation (21) in equation (20), to express the Fourier law in terms of the luminosity:

$$\frac{L}{4\pi r^2} = -\kappa \frac{\partial T}{\partial r}. \quad (22)$$

Furthermore the heat conductivity needs to be expressed in terms of physical parameters. In that case one must first look at two cases, the short mean free path (SMFP) and the long mean free path (LMFP) regime. In order to quantify both cases first the gravitational scale H height and the Knudsen number Kn need to be introduced. The gravitational scale height is defined as

$$H := \sqrt{\frac{\nu^2}{4\pi G\rho}} \quad (23)$$

and the Knudsen number is defined as the ratio of the mean free path Λ and the gravitational scale height

$$Kn := \frac{\Lambda}{H}. \quad (24)$$

In the SMFP regime the scale height is much larger than the mean free path or equivalently $Kn \ll 1$. If the core of the halo is in this regime, particle motion is heavily constrained by collisions; this implies that the heat transfer from the core to the extended halo, which is the driving force behind the gravothermal collapse as explained in section 2.2, is limited to the outermost surface of the core. On the contrary, in the LMFP regime the mean free path is much larger than the scale height or equivalently $Kn \gg 1$. In this regime individual particles can orbit the center multiple times without being scattered. Due to the large mean free paths, particles may escape the core of the halo, when scattered only once. This implies that the entire core can contribute to the heat and mass transfer[3].

The heat conductivity in the SMFP regime can be calculated using first order Chapman-Enskog theory (CE). In CE the distribution function and all transport quantities are developed into an infinite series, where the zeroth term is the quantity in equilibrium. In the case of the heat flux the expansion becomes[11]

$$\mathbf{q} = \sum_{n=0}^{\infty} \mathbf{q}^{(n)}. \quad (25)$$

First observe that the heat flux in equilibrium $\mathbf{q}^{(0)} = 0$. When considering only terms up to first order, equation (25) simply reduces to $\mathbf{q} = \mathbf{q}^{(1)}$ [11]. Now using CE, one quickly recovers equation (20) and furthermore obtains an explicit form for the heat conductivity in the SMFP regime κ_{SMFP} ,

$$\kappa_{\text{SMFP}} = \frac{3k_B}{2m} b \rho \frac{\Lambda^2}{\tau}, \quad (26)$$

where $b = 25\sqrt{\pi}/32$ can be interpreted as an impact parameter and $\tau := \Lambda/\nu$ is the time between collisions.

The heat conductivity in the LMFP regime κ_{LMFP} can not be calculated from first principles, since CE requires that the time between collisions is short compared to macroscopic variations of the fluid's properties. To obtain an expression for the heat conductivity in the LMFP regime, the mean free path is replaced by the scale height and the collision time is replaced by the relaxation time $t_r := 1/(a\rho\sigma_m\nu)$, where $a = 4/\sqrt{\pi}$, for hard sphere scattering and σ_m is the cross section per unit mass. Due to the lack of a rigorous derivation of κ_{LMFP} , b is replaced by a new parameter C^6 , which needs to be determined by fitting the model to N -body simulations. The resulting expression for the LMFP heat conductivity is

$$\kappa_{\text{LMFP}} = \frac{3k_B}{2m} C \rho \frac{H^2}{t_r}. \quad (27)$$

The SMFP and LMFP regimes only cover the two limiting cases for $Kn \ll 1$ or $Kn \gg 1$. To obtain a general expression for the heat conductivity, one must interpolate between these two regimes. An early approach was to use, up to a factor of 2, the harmonic mean of κ_{SMFP} and κ_{LMFP} [3],

⁶Sometimes C is also called β in literature[26], [13].

$$\kappa = \left(\kappa_{\text{SMFP}}^{-1} + \kappa_{\text{LMFP}}^{-1} \right)^{-1}. \quad (28)$$

However it was noted in [45], [29], that this approach can not adequately capture the transition between the SMFP and LMFP regimes. So Nishikawa and his colleagues proposed a more general form of the interpolation, introducing a second free parameter α in the model, which analogously to the parameter C needs to be determined by comparison to N -body simulations. Now instead of the harmonic mean, the power mean is used for the interpolation and the heat conductivity takes the form[29]

$$\kappa = \left(\kappa_{\text{SMFP}}^{-\alpha} + \kappa_{\text{LMFP}}^{-\alpha} \right)^{-\frac{1}{\alpha}}. \quad (29)$$

Using equations (26) and (27), as well as the definition of the velocity dispersion, equation (16) and further definitions given in this section, equation (22) can be expressed as

$$\boxed{\frac{L}{4\pi r^2} = -\frac{3}{2}ab\nu\sigma_m \left((a\sigma_m^2)^\alpha + \left(\frac{4\pi Gb}{C\rho\nu^2} \right)^\alpha \right)^{-\frac{1}{\alpha}} \frac{\partial \nu^2}{\partial r}}. \quad (30)$$

2.2.5 Laws of thermodynamics

The fourth and final fundamental equation comes from the laws of thermodynamics. Under the assumption of a quasistatic process[3], [17] the second law of thermodynamics relates heat transfer dQ to the change in entropy dS as

$$TdS = dQ. \quad (31)$$

Expressing the second law with the specific entropy s yields,

$$T \left(\frac{\partial s}{\partial t} \right)_M = \frac{1}{\rho V} \left(\frac{dQ}{dt} \right)_M = -\frac{1}{\rho} \nabla \cdot \mathbf{q}, \quad (32)$$

where $\mathbf{q}$ is the outgoing heat flux. Inserting the luminosity and using spherical symmetry, one obtains

$$T \left(\frac{\partial s}{\partial t} \right)_M = -\frac{1}{4\pi\rho r^2} \frac{\partial L}{\partial r}. \quad (33)$$

Using the first law of thermodynamics next and rewriting it for the specific thermodynamic quantities results in

$$dU = TdS - pdV \Leftrightarrow du = Tds + \frac{p}{\rho^2} d\rho, \quad (34)$$

where U is the internal energy and u is the specific integral energy. Using the chain rule to expand du as $du = \left(\frac{\partial u}{\partial T} \right)_p dT + \left(\frac{\partial u}{\partial p} \right)_T dp$ and solving for ds yields

$$ds = \frac{1}{T} \left(\frac{\partial u}{\partial T} \right)_p dT + \frac{1}{T} \left(\frac{\partial u}{\partial p} \right)_T dp - \frac{p}{T\rho^2} d\rho. \quad (35)$$

Using that $u = \frac{3}{2} \frac{k_B T}{m}$, which implies $\left(\frac{\partial u}{\partial p} \right)_T = 0$, one finds for the above expression

$$ds = \frac{k_B}{m} \left(\frac{3}{2} \frac{dT}{T} - \frac{d\rho}{\rho} \right), \quad (36)$$

after integrating this equation, one obtains an expression for the specific entropy,

$$s = \frac{k_B}{m} \ln \left(\frac{T^{\frac{3}{2}}}{\rho} \right). \quad (37)$$

The new found expression for the specific entropy can be inserted into equation (33), after solving (33) for $\partial L / \partial r$ and replacing T with ν , via equation (16), one obtains the final fundamental equation of the gravothermal model. It can be explicitly written as

$$\boxed{\frac{\partial L}{\partial r} = -4\pi r^2 \rho \nu^2 \left(\frac{\partial}{\partial t} \right)_M \ln \left(\frac{\nu^3}{\rho} \right)}. \quad (38)$$

3 Numerical methods

3.1 Numerical implementation of the gravothermal fluid model

In section 2.2 the equations governing the gravothermal collapse in the fluid approximation were derived, see equations (4), (19), (30) and (38). To integrate these equations adequately one needs to turn them into a dimensionless form first. For that, dimensionless variables, indicated by a tilde, are introduced. To obtain a dimensionless quantity, a dimensional quantity needs to be divided by a given scale; the corresponding scales are denoted by a *zero* in the subscript. The scales for r , ρ , t , M , ν , σ_m and L are given in table 1. The scale factors for radius r_s and density ρ_s , as halo specific properties, need to be given as initial conditions.

Table 1: Table denoting the expression for the scale factors of the radius r_0 , the density ρ_0 , the time t_0 , the mass M_0 , the velocity dispersion ν_0 , the cross section per unit mass $\sigma_{m,0}$ and the luminosity L_0 [6].

r_0	ρ_0	t_0	M_0	ν_0	$\sigma_{m,0}$	L_0
r_s	ρ_s	$\frac{1}{\sqrt{4\pi G \rho_s}}$	$4\pi r_s^3 \rho_s$	$\sqrt{\frac{GM_0}{r_s}}$	$\frac{4\pi r_s^2}{M_0}$	$\frac{GM_0^2}{r_s t_0}$

With the introduction of dimensionless quantities, the equations governing the gravothermal collapse can be written as[29]

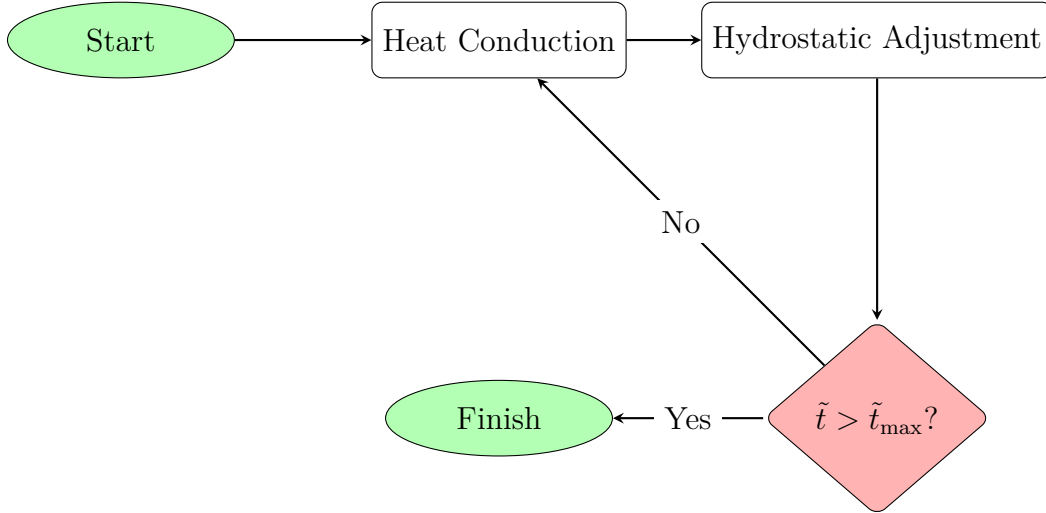


Figure 2: Flow chart diagram of the numerical implementation of the gravothermal fluid model.

$$\left\{ \begin{array}{l} \frac{\partial \tilde{M}}{\partial \tilde{r}} = \tilde{\rho} \tilde{r}^2 \\ \frac{\partial (\tilde{\rho} \tilde{v}^2)}{\partial \tilde{r}} = -\frac{\tilde{M} \tilde{\rho}}{\tilde{r}^2} \\ \tilde{L} = -\frac{3}{2} \tilde{r}^2 a b \tilde{v} \tilde{\sigma}_m \left((a \tilde{\sigma}_m)^\alpha + \left(\frac{b}{C \tilde{\rho} \tilde{v}^2} \right)^\alpha \right)^{-\frac{1}{\alpha}} \frac{\partial \tilde{v}^2}{\partial \tilde{r}} \\ \frac{\partial \tilde{L}}{\partial \tilde{r}} = -\tilde{r}^2 \tilde{\rho} \tilde{v}^2 \left(\frac{\partial}{\partial \tilde{t}} \right)_{\tilde{M}} \ln \left(\frac{\tilde{v}^3}{\tilde{\rho}} \right) \end{array} \right. \quad (39)$$

The halo is discretized into N concentric shells, which are initially logarithmically spaced at radii $\tilde{r}_1, \dots, \tilde{r}_N$. It is important to note that extensive quantities evaluated at the i -th shell, like $\tilde{M}_i, \tilde{L}_i$, denote the total value enclosed at the dimensionless radius $\tilde{r}_i$. Meanwhile intensive quantities evaluated at the i -th shell, like $\tilde{p}_i, \tilde{\rho}_i$ are evaluated at the midpoint of the i -th shell, so at radius $(\tilde{r}_i + \tilde{r}_{i-1})/2$.

The principle idea behind the evolution is that a heat conduction step pushes the halo out of hydrostatic equilibrium. This is followed by hydrostatic adjustment steps, which restore the hydrostatic equilibrium. After a heat conduction step the time is advanced [29]. The halo evolution ends after the dimensionless time exceeds a maximum time $\tilde{t}_{\max}$. A flow chart representation can be found in figure 2.

To be more specific the heat conduction changes the energy density in a shell by $\delta \tilde{u}$. Using equation (38), the change in specific energy can be quantified as [29]

$$\delta \tilde{u} = -\frac{\partial \tilde{L}}{\partial \tilde{M}} \delta \tilde{t}, \quad (40)$$

where $\delta \tilde{t}$ is the time the heat conduction step takes. It is assumed that $\delta \tilde{t}$ is much smaller than the relaxation time in any shell of the halo $t_{r,i}$, so that the mass is constant in a given a shell, i.e. $\delta \tilde{t} := \epsilon_t \min_{i \in 0, \dots, N} (t_{r,i}/t_0)$, with $\epsilon_t \ll 1$. The change in specific energy in

a given shell $\delta\tilde{u}_i$, induces a change a pressure via the relation $\delta\tilde{p}_i = \frac{2}{3}\tilde{\rho}_i\delta\tilde{u}_i$, which pushes the shell out of hydrostatic equilibrium[29]. This means that the hydrostatic equilibrium condition, equation (19), here written in a discretized form, no longer applies[33]

$$\frac{\tilde{p}_{i+1} - \tilde{p}_i}{(\tilde{r}_{i+1} - \tilde{r}_{i-1})/2} + \frac{\tilde{\rho}_{i+1} + \tilde{\rho}_i}{2} \frac{\tilde{M}_i}{\tilde{r}_i^2} \neq 0. \quad (41)$$

To return to hydrostatic equilibrium, the pressures and densities in every shell, as well as the outer radius of the shell, need to be adjusted, i.e. $\tilde{p}_i \rightarrow \tilde{p}_i + \delta\tilde{p}_i$, $\tilde{\rho}_i \rightarrow \tilde{\rho}_i + \delta\tilde{\rho}_i$ and $\tilde{r}_i \rightarrow \tilde{r}_i + \delta\tilde{r}_i$. Such that the hydrostatic equilibrium condition is satisfied again, i.e[33].

$$\frac{\tilde{p}_{i+1} + \delta\tilde{p}_{i+1} - \tilde{p}_i - \delta\tilde{p}_i}{(\tilde{r}_{i+1} + \delta\tilde{r}_{i+1} - \tilde{r}_{i-1} - \delta\tilde{r}_{i-1})/2} + \frac{\tilde{\rho}_{i+1} + \delta\tilde{\rho}_{i+1} + \tilde{\rho}_i + \delta\tilde{\rho}_i}{2} \frac{\tilde{M}_i}{\tilde{r}_i^2 + 2\tilde{r}_i\delta\tilde{r}_i} = 0. \quad (42)$$

Note that the adjustments are assumed to be small, such that they are only considered in linear order. Next the corrections in pressure $\delta\tilde{p}_i$ and in density $\delta\tilde{\rho}_i$, need to be expressed in terms of the radial correction $\delta\tilde{r}_i$. From the conservation of mass follows[33]

$$\delta\tilde{M}_i = \tilde{\rho}_i\delta\tilde{V}_i + \tilde{V}_i\delta\tilde{\rho}_i = 0 \Leftrightarrow \delta\tilde{\rho}_i = -\tilde{\rho}_i \frac{\delta\tilde{V}_i}{\tilde{V}_i}, \quad (43)$$

where $\tilde{V}_i$ denotes the volume of the i -th shell. Since the hydrostatic adjustment is adiabatic, the quantity $\tilde{A}_i := \tilde{p}_i\tilde{V}_i^{\frac{5}{3}}$ is conserved. This implies[33]

$$\delta\tilde{A}_i = \tilde{V}_i^{\frac{5}{3}}\delta\tilde{p}_i + \frac{5}{3}\tilde{p}_i\tilde{V}_i^{\frac{2}{3}}\delta\tilde{V}_i = 0 \Leftrightarrow \delta\tilde{p}_i = -\frac{5}{3}\tilde{p}_i \frac{\delta\tilde{V}_i}{\tilde{V}_i}. \quad (44)$$

Since the shells are spherical, the volume is easily calculated and from there the adjustment as a function of radius can be obtained as[33]

$$\tilde{V}_i = \frac{4\pi}{3} (\tilde{r}_i^3 - \tilde{r}_{i-1}^3) \implies \delta\tilde{V}_i = 4\pi (\tilde{r}_i^2\delta\tilde{r}_i - \tilde{r}_{i-1}^2\delta\tilde{r}_{i-1}). \quad (45)$$

Inserting equation (45) into equations (43) and (44), one obtains for the pressure and density adjustments[33]

$$\begin{cases} \delta\tilde{\rho}_i = -3\tilde{\rho}_i \frac{\tilde{r}_i^2\delta\tilde{r}_i - \tilde{r}_{i-1}^2\delta\tilde{r}_{i-1}}{\tilde{r}_i^3 - \tilde{r}_{i-1}^3} \\ \delta\tilde{p}_i = -5\tilde{p}_i \frac{\tilde{r}_i^2\delta\tilde{r}_i - \tilde{r}_{i-1}^2\delta\tilde{r}_{i-1}}{\tilde{r}_i^3 - \tilde{r}_{i-1}^3} \end{cases}. \quad (46)$$

Inserting equation (46) back into (42) and linearizing the denominators, a tridiagonal system of linear equations[29]

$$a_i\delta\tilde{r}_{i-1} + b_i\delta\tilde{r}_i + c_i\delta\tilde{r}_{i+1} = d_i, \quad \text{for } i \in \{1, \dots, N-1\}, \quad (47)$$

is obtained. a_i , b_i , c_i and d_i are constants depending on pressure, mass and density. With the boundary conditions $\tilde{r}_0 = \delta\tilde{r}_0 = 0 = \delta\tilde{r}_N$ and $L_0 = 0$ the linear system of equations (47) can be solved. After solving this linear system of equations all other quantities are updated as well. Since it is assumed that the time it takes the halo to return to hydrostatic equilibrium is much smaller than the relaxation time, so that the time is not updated after a hydrostatic adjustment step. Before the next heat conduction step, the hydrostatic adjustment step is repeated until $\max_{0 < i < N} \left| \frac{\delta\tilde{r}_i}{\tilde{r}_i} \right| < \epsilon_r$, with $\epsilon_r \ll 1$ [29].

3.2 N -body simulations

N -body simulations have been widely used to understand the properties and the evolution of SIDM halos. So there exist many different N -body codes [44]. The code used for the N -body simulations in this thesis is the OpenGadget3. This section is based on a description given section in 2.1. of [16]. The principle idea is that instead of simulating individual particles, the phase space density f is considered. The justification is that a very large number of dark matter particles are described, allowing the phase space density to be well defined. The evolution of f is determined by the Boltzmann equation. In an astrophysical context gravity is the predominant force, describing the evolution of the phase space density. Thus the Boltzmann equation becomes⁷

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f - \nabla_{\mathbf{x}} \phi \cdot \nabla_{\mathbf{v}} f = \left(\frac{\partial f}{\partial t} \right)_{\text{coll}}, \quad (48)$$

where ϕ is the gravitational potential and $\left(\frac{\partial f}{\partial t} \right)_{\text{coll}}$ is the so-called collision term, which is used to describe the self-interactions. In numerical computations the phase space density needs to be discretized into N numerical particles, each having a position $\mathbf{x}$, a velocity $\mathbf{v}$, a mass m_N and a kernel. Each numerical particle represents a phase space patch of physical particles at position $\mathbf{x}$ and all having velocity $\mathbf{v}$. The kernel is used to calculate the density of a numerical particle, which is used for the modelling of self-interactions. A detailed overview of all features of OpenGadget3 can be found in [16].

3.3 Nested sampling

3.3.1 Bayesian inference

The fundamental question of Bayesian inference is that given a set of data D , one wants to infer on a model M , which depends on parameters Θ_M . So the goal is to quantify the probability that the parameters Θ_M are given with data D , the so-called *posterior* $\mathcal{P}(\Theta) := P(\Theta_M | D, M)$. However only the *likelihood* $\mathcal{L}(\Theta) := P(D | \Theta_M, M)$, i.e. the probability that given a model with parameters Θ_M the data D is obtained, is accessible directly. The likelihood and the posterior can be related through Bayes theorem [39]

$$P(\Theta_M | D, M) = \frac{P(D | \Theta_M, M) P(\Theta_M, M)}{P(D, M)} \Leftrightarrow \mathcal{P}(\Theta) = \frac{\mathcal{L}(\Theta) \pi(\Theta)}{\mathcal{Z}}, \quad (49)$$

where $\pi(\Theta) := P(\Theta_M, M)$ denotes the prior belief on the model M without any data, based on physical constraints and is therefore called *prior*. So the posterior can be seen as

⁷The equation is also known as the Vlasov-Poisson equation [16].

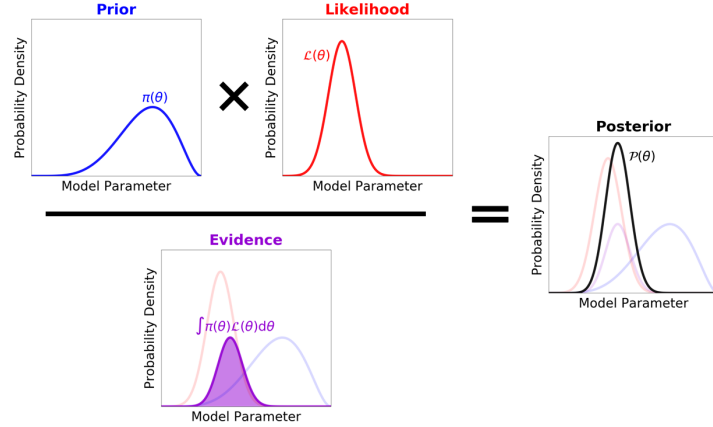


Figure 3: Illustration of Bayes theorem, equation (49), reprinted from[39].

the belief in the model after combining prior belief with data. $\mathcal{Z} := P(D, M)$ is called *evidence* and can be related to the posterior and the likelihood as[39]

$$\mathcal{Z} := P(D, M) = \int d\Theta \mathcal{L}(\Theta) \pi(\Theta), \quad (50)$$

since the evidence is a normalization constant. A graphical illustration of Bayes theorem can be found in figure 3.

An important concept is the posterior mass. The quantity $\mathcal{P}(\Theta)$ describes a probability density, not the total probability, meaning that if $\mathcal{P}(\Theta)$ has a large peak but the volume in parameter space, where the peak is sufficiently large is small, the total probability is also small. On the other hand, a smaller peak, which is extended over a larger volume, can be more likely[22]. The integration of the posterior density over a volume of parameter space, i.e. the total probability, is called posterior mass[39]. Methods like nested sampling do not find the global maximum of the posterior density but instead the largest concentration of posterior mass[39], [22].

3.3.2 Standard nested sampling

The philosophy behind nested sampling is to split a difficult problem, sampling from a posterior, into many easier problems by slicing up the posterior into simpler distributions, then sampling from these easier distributions and finally combining them, with proper weights, to calculate the evidence and obtain the posterior[40].

According to equation (50), the evidence is calculated with an integral over the product of prior and likelihood. This integral over a d -dimensional parameter space Ω_Θ can be rewritten into the one dimensional integral[40]

$$\mathcal{Z} = \int_{\Omega_\Theta} d\Theta \mathcal{L}(\Theta) \pi(\Theta) = \int_0^1 dX \mathcal{L}(X), \quad (51)$$

where $\mathcal{L}(X)$ is the edge of the volume X , which is formally defined as[40]

$$X(\eta) := \int_{\mathcal{L}(\Theta) \geq \eta} d\Theta \cdot \pi(\Theta). \quad (52)$$

So X denotes the region of the prior, where the likelihood is larger than η . Note that $X = 1$ is the entire prior volume, while $X = 0$ is the maximum of the likelihood. The evidence in the form of equation (51) can be calculated numerically by approximating it with N discrete points, turning the continuous integral into a discrete sum[39]

$$\mathcal{Z} \approx \sum_{i=1}^N \frac{1}{2} (\mathcal{L}(\Theta_i) + \mathcal{L}(\Theta_{i-1})) \cdot (X_i - X_{i-1}) =: \sum_{i=1}^N \omega(\Theta_i), \quad (53)$$

where $X_0 = 1$ and $\omega_i := \frac{1}{2} (\mathcal{L}(\Theta_i) + \mathcal{L}(\Theta_{i-1})) \cdot (X_i - X_{i-1})$ is called importance weight. With the weights the posterior can be approximated as[40]

$$\mathcal{P}(\Theta) \approx \frac{\sum_{i=1}^N \omega_i(\Theta_i) \delta(\Theta_i)}{\mathcal{Z}}, \quad (54)$$

where $\delta(\Theta_i)$ is the delta distribution evaluated at Θ_i . The objective of nested sampling is to find the volume X as a function of η . In general it is difficult to sample from a raw prior, since there can be strong variations of the probability density over the considered parameter space. So the prior is transformed, with a suitable transformation, transformed into a d -dimensional unit cube, where d is the dimension of the parameter space. To estimate changes in volume, one first draws a set of K^8 points called *live points* from the transformed prior. Following that, the main loop of the nested sampling algorithm begins. After each iteration of the loop, the point with the lowest likelihood is added to the set of *dead points* and is replaced by another live point with a larger likelihood, such that the total number of live points K is a constant. So the volume X_i at the i -th iteration, corresponding to a set of i dead points, can be related to the volume at the $i - 1$ iteration with[19]

$$X_i = t_i X_{i-1} = \prod_{j=0}^i t_j, \quad (55)$$

where t_i is the shrinkage ratio. Although each t_i is a random number between 0 and 1, one still can make deterministic assertions about the expectation value[40],

$$\mathbb{E}(X_i) \approx \left(\frac{K}{K+1} \right)^i \Leftrightarrow \mathbb{E}(\ln(X_i)) \approx -\frac{i}{K}. \quad (56)$$

An important implication is that is that the logarithmic step size of the volume is constant and especially fully independent of the value of the likelihood, i.e.[40]

$$\mathbb{E}(\ln(X_i)) - \mathbb{E}(\ln(X_{i-1})) \approx \frac{1}{K}. \quad (57)$$

⁸It is important to note, that a larger number of live points gives a more accurate evaluation of the volume[40].

From there the value of the evidence can be obtained, with equation (53). If the value of the value for the evidence converges, i.e. if at the i -th iteration[40]

$$\Delta \ln(\mathcal{Z}_i) := \ln(\mathcal{Z}_i + \Delta \mathcal{Z}_i) - \ln(\mathcal{Z}_i) < \epsilon, \quad (58)$$

with ϵ a small positive number, then the main loop stops. To improve the final result for the posterior and evidence, the remaining live points are added as dead points.

3.3.3 Dynamic nested sampling

The main difference between DNS and standard nested sampling is that the number of live points can vary over the course of a DNS run. While the primary focus of standard nested sampling is the evidence, the posterior is only obtained as a byproduct. The variation of live points can allow the focus to shift from the evidence to the posterior, which is the interesting quantity for Bayesian inference.

The inflexibility of standard nested sampling is encoded in equation (57). As can be seen from equation (54), the posterior depends on the difference of two volumes, multiplied by the likelihood. If this difference is not adjusted to the likelihood, as is the case in standard nested sampling, see equation (57), then all parts of the likelihood, independent of their contribution to the posterior, are treated the same. On the other hand, if a variable number of live points is used, then one can choose a smaller volume difference, where the contribution is sufficiently large, to resolve the posterior better. Whereas in a static nested sampling case the overall number of live points, so also the runtime, would be needed to increased[40].

4 Implementation

4.1 Implementation of the generalized heat conductivity

The code used to perform the simulations in the gravothermal fluid model is called `GravothermalSIDM`[6], [31], [18]. However in the code only the more restricted heat conductivity model, equation (28), has been implemented, instead of the generalized one, equation (29). So the implemented heat conductivity interpolation model needed to be replaced by the more general model.

Doing so requires two steps. First the parameter α needs to be introduced and second the line where the heat conductivity is calculated needs to be changed, such that the generalized heat conductivity model is used.

Each simulated run is an instance of the `Halo` class. All parameters relevant for the halo evolution in the gravothermal model are attributes of that class, so it is natural to implement α as a new attribute `alpha` of the `halo` class.

In order to implement the generalized heat conductivity, the line of the code that calculates the heat conductivity needs to be modified, such that the generalized heat conductivity model is used instead of the older, more restricted one. The changed line of code is shown in listing 1, where the modifications to the `GravothermalSIDM` code are shown in red.

```

...
Keff = 1./(self.Kinv_smfp**self.alpha + self.Kinv_lmfp**self.alpha)
** (1./self.alpha)
...

```

Listing 1: Implementation of the interpolation of the heat conductivity K_{eff} in the `GravothermalSIDM` code. `Kinv_smfp` and `Kinv_lmfp` denote the inverses of the short- and long mean free path heat conductivity respectively. The parts that are marked red are modifications to the original code.

4.2 Simulation setup

In this thesis four different halos were studied. They can be characterized by their initial profile and cross section per unit mass, since SIDM has non-negligible self-interactions; the cross section, quantifying these interactions, is also required to describe the halo. For all halos it is assumed that the initial density profile follows the NFW-profile, equation (1). So, as explained in section 2.1, only the mass and concentration⁹ are necessary to describe the halo profile initially. Two halos only differ in cross section per unit mass; this still means, that the evolution differs and thus they are treated as two distinct halos. In this thesis a constant cross section is studied. A more general and realistic class of cross sections are velocity-dependent cross sections, where $\sigma_m = \sigma_m(v)$ [44]. So to summarize, the mass, concentration and cross section determine the halo fully.

A parameter, which is not a physical property of the halos, but important for the N -body simulations and relevant for this thesis, is the number of numerical particles used to conduct the N -body simulations. All of these parameters are given in table 2.

Table 2: Table denoting the physical parameters describing the halo, as well as the number of numerical particles N used in the corresponding N -body simulation for all halos considered. The parameters are taken from [15]. All numerical values are rounded to the second significant digit. The naming convention for the studied halos follows the one introduced in [45].

Name	Halo1sig30	Halo1sig80	Halo2sig80	Halo3sig700
σ_m [cm ² /g]	30	80	80	700
c	22	22	26	8
M [$10^{10} M_\odot$]	0.28	0.28	$3.24 \cdot 10^{-3}$	$5.77 \cdot 10^{-3}$
N	10^7	$5 \cdot 10^7$	$2 \cdot 10^6$	$2 \cdot 10^6$

The N -body simulations were done by Moritz Fischer. The simulations using the modified `GravothermalSIDM` code were performed by the author. As can be seen from figure 2 the official termination criterion for the fluid simulations was that the dimensionless time is larger than a given time value $\tilde{t}_{\text{max}}$. Since data from the fluid simulations will be compared against N -body simulation data, an easy choice for $\tilde{t}_{\text{max}}$ is the dimensionless time of the latest snapshot of the N -body simulations¹⁰. However some fluid simulations

⁹Alternatively the scale radii and densities can be used. They are given in table 3.

¹⁰A snapshot saves all relevant information about a halo at a specific time.

collapse before $\tilde{t}_{\max}$ is reached, afterwards time progression within the fluid model becomes impossible and the simulations were stopped. The implications of this are discussed in section 4.3.3. The physical parameters used for the gravothermal simulations are given in table 3.

Table 3: Table denoting the simulation parameters used for the fluid simulations. r_s and ρ_s are again taken from [15]. All numerical values are rounded to the second significant digit. Some simulations collapsed before $\tilde{t}_{\max}$ was reached. After the collapse the simulations were stopped, although $\tilde{t}_{\max}$ was not reached. All remaining physical parameters, except for α and C , are the default parameters used in the `GravothermalSIDM` code.

Name	Halo1sig30	Halo1sig80	Halo2sig80	Halo3sig700
r_s [kpc]	1.28	1.28	0.25	0.82
ρ_s [$M_\odot/\text{pc}^3$]	$4.42 \cdot 10^{-2}$	$4.42 \cdot 10^{-2}$	$6.83 \cdot 10^{-2}$	$3.55 \cdot 10^{-3}$
σ_m [cm^2/g]	30	80	80	700
n_{shells}	200	200	200	200
$r_{\min}$ [kpc]	10^{-3}	10^{-3}	$2.51 \cdot 10^{-4}$	$8.15 \cdot 10^{-4}$
$r_{\max}$ [kpc]	19.24	19.24	19.2	19.2
$\tilde{t}_{\max}$	255	545	660	470

4.3 Fitting method

As stated in section 1 the objective of this thesis is to determine the free parameters α and C in the generalized heat conductivity model, equation (29). To accomplish that, the evolution of a halo in the gravothermal fluid model must be compared to the halo evolution in a N -body simulation. Since the concept of a heat conductivity does not exist in N -body simulations, the heat conductivity can not be fitted directly, but the free parameters must be determined indirectly by comparing other observables both methods have in common.

In this thesis the enclosed number of numerical particles¹¹ within a radius r were compared. As can be easily verified from section 2.2 the fluid model does not have numerical particles, since it treats matter as a continuous fluids. However one can make use that the number of numerical particles within a radius r at a given time t can be related to the total enclosed mass M_N via

$$N = \frac{M_N}{m_N}, \quad (59)$$

where m_N is the mass of a numerical particle. Though the enclosed mass and the enclosed particle number at radius r and time t are formally functions of t and r , their t and r dependence, as well as every other t and r dependence is suppressed for sake of readability in this section. An exception is section 4.3.8. Since the enclosed mass at a given radius is an observable in the gravothermal fluid model as well, see equation (4), an expected number of numerical particles λ can be defined as

¹¹The concept is explained in section 3.2.

$$\lambda(\alpha, C) := \frac{M(\alpha, C)}{m_N}, \quad (60)$$

where $M(\alpha, C)$ is the enclosed mass in the gravothermal fluid model. Note that in contrary to a physical particle number, which is limited to the natural numbers, equation (60) allows λ in principle to take any positive real value.

Physically the parameters α and C can take continuous values. Since a simulation of the gravothermal collapse in the fluid model is still numerically expensive¹², an evaluation of λ at an arbitrary parameter pair α and C should not include a simulation of the gravothermal collapse. In order to still obtain a value λ at these parameters one needs to resort to interpolation. So first simulations were run for a discrete set of parameter pairs α and C , so for these parameter pairs λ is known for all radii and times. For each radius and time an interpolating function can now be constructed in a specified region of parameter space. This allows for a comparison against the number of enclosed numerical particles from N -body simulations. To be more concrete, the number of enclosed numerical particles should be compared to the enclosed number of expected numerical particles at a specific radius, but for all times. From there a suitable likelihood and prior can be constructed and with using DNS the posterior can be obtained. With a known posterior the best fit values for α and C at one radius can be determined. Finally from the best fit values at the individual radii, the halo best fit can be obtained.

The choice of the studied parameter region is explained in section 4.3.1. Section 4.3.2 explains the choice of simulated fluid runs to construct the interpolation function for the studied parameter region. Section 4.3.3 explains the interpolation method used to construct an interpolating function $\lambda(\alpha, C)$. Section 4.3.4 and section 4.3.5 illustrate the choice of prior and likelihood respectively. Section 4.3.6 describes the implementation of DNS in this thesis and finally section 4.3.7 explains how to the best fit for the entire halo is obtained from the best fits at the individual radii. An alternative fitting method is outlined in section 4.3.8. A schematic illustration of the fitting scheme is given in figure 4.

¹²A realistic simulation may take days, but this is strongly dependent on the initial conditions and the computer used to run the simulation on.

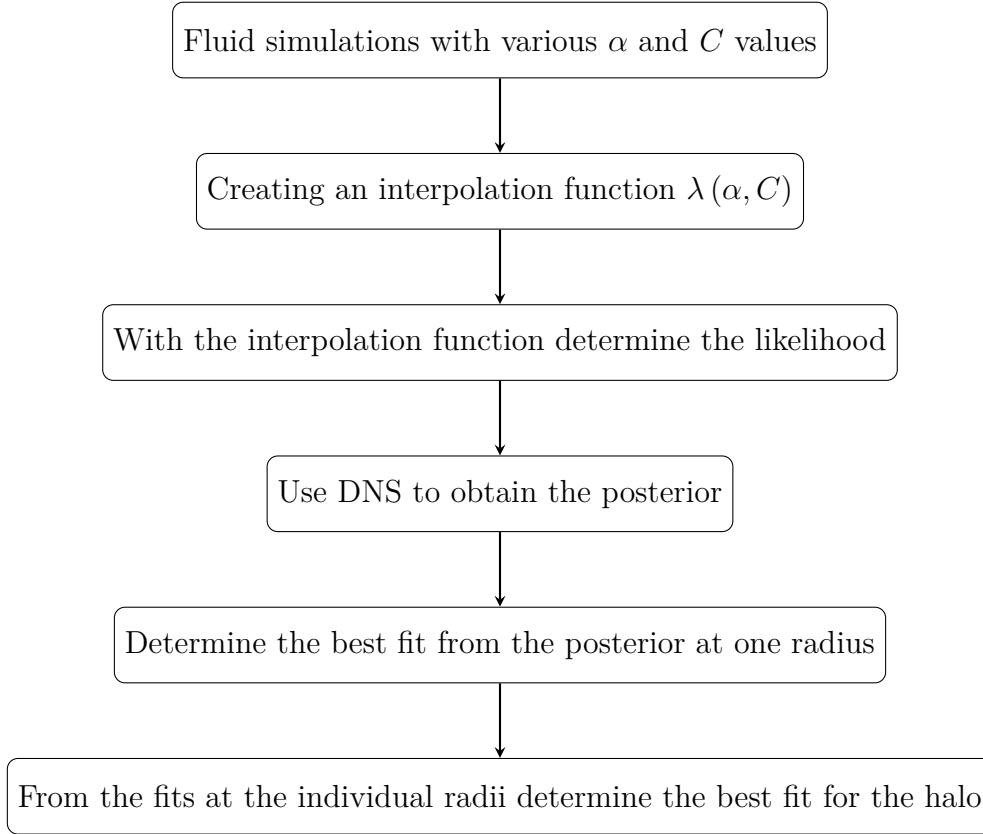


Figure 4: Flow chart diagram of the structure of the implementation.

The code used to conduct the analysis can be found here: <https://github.com/kianfouladi-gif/Bachelor-Thesis-code>. Parts of the code originally are based on [46], but were rewritten for this thesis.

4.3.1 Constraints on parameter space

The only physical constraints on α and C are that they are both positive real numbers, i.e. $\alpha, C > 0$. If $\alpha < 0$ the interpolation for the heat conductivity, equation (29), would reduce the heat conductivity in the SMFP regime in the LMFP limit and vice versa and if $\alpha = 0$, equation (29) would include a $1/0$ term and is hence undefined. If $C \leq 0$ the heat conductivity in the LMFP regime would be negative or identical zero, which is unphysical. Since it is not feasible to construct an interpolation function over an infinitely large part of parameter space, further constraints need to be given on the parameters α and C . Since C was previously extensively studied in the context of the simplified heat conductivity interpolation with only C as a free parameter, equation (28), constraints on C already exist. It is assumed by many authors that $C \in [0.6, 1]$ [26], [13], [21]. Since this only holds for the one parameter model and in this thesis a second free parameter is considered, the bounds of the interval considered were extended by 0.2 in both directions, so $C \in [0.4, 1.2]$ is considered. The literature regarding the parameter α is far more sparse, but similar studies suggest $\alpha \lesssim 1$. One peculiar property is that the effect of the parameter α seems to be irrelevant for the evolution of an halo for $\alpha > 1$ [27], [45]. This is illustrated in figure 5, where the expected number of enclosed numerical particles at a radius of 0.1 kpc is shown as function of time for all halos considered, except Halo2sig80 and at a radius of

0.01 kpc for Halo2sig80. All of the runs simulated with the fluid model have $C = 0.6$ and varying values of α between 0.2 and 4.

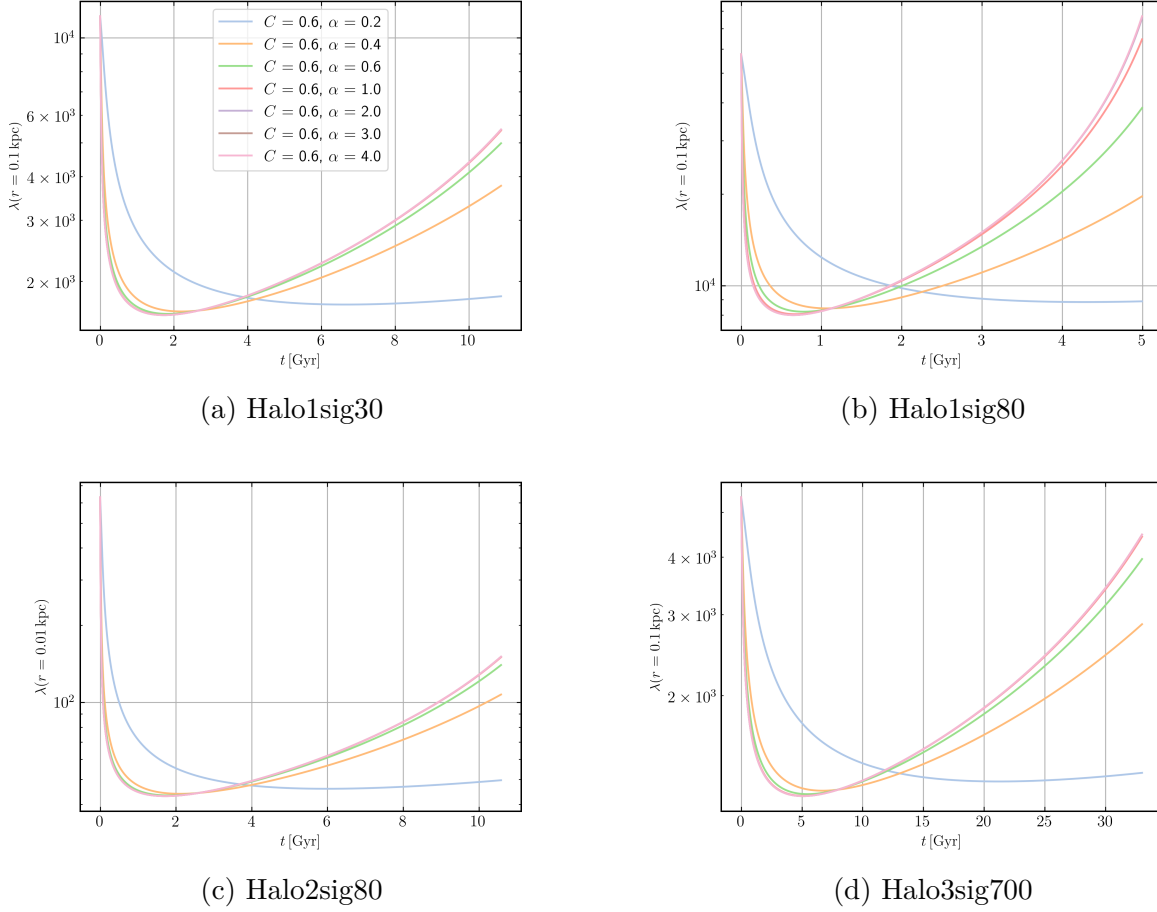


Figure 5: The expected number of enclosed numerical particles is shown as function of time. a.) shows it for Halo1sig30 at a radius of 0.1 kpc, b.) for Halo1sig80 at a radius of 0.1 kpc, c.) for Halo2sig80 at a radius of 0.01 kpc and d.) for Halo3sig700 at a radius of 0.1 kpc. The parameters chosen for the fluid simulation were $C = 0.6$ and $\alpha = 0.2, 0.4, 0.6, 1, 2, 3, 4$.

As can be seen from figure 5, the cases $\alpha = 1, 2, 3, 4$ can barely be distinguished. Since only the heat conductivity depends on α , the influence of α on the halo evolution can be fully characterized by computing the derivative of the dimensionless effective heat conductivity¹³ $\tilde{K}$ with respect to α ,

$$\frac{d\tilde{K}}{d\alpha} \approx \frac{\tilde{K}(\alpha + \Delta\alpha) - \tilde{K}(\alpha)}{\Delta\alpha}. \quad (61)$$

Note that a finite difference approximation needs to be used for the calculation instead of a symbolic differentiation, since the heat conductivity depends on quantities like density or

¹³The effective heat conductivity is defined as $K := \frac{2}{3}\kappa m$, where κ is the heat conductivity and m is the mass of the particle[6].

velocity dispersion that are themselves functions of α . Using $\Delta\alpha = 0.2$, one finds for the absolute value of $\frac{d\tilde{K}}{d\alpha}$ averaged over all radial and temporal bins, $\left|\overline{\frac{d\tilde{K}}{d\alpha}}\right| = \mathcal{O}(10^{-1} - 10^{-2})$ for $\alpha = 0.2$ and $\left|\overline{\frac{d\tilde{K}}{d\alpha}}\right| = \mathcal{O}(10^{-2} - 10^{-3})$ for $\alpha = 1$. The bar denotes the arithmetic mean. So $\left|\overline{\frac{d\tilde{K}}{d\alpha}}\right|$ drops by one order of magnitude, when α increases from 0.2 to 1. This implies that the effect of the parameter α diminishes for $\alpha > 1$, as can also be seen in figure 5.

This behavior can be used to set an upper bound for α . Since the halo evolution seems to be independent of α for $\alpha \gtrsim 1$, only α values smaller than or equal to 1.2 are considered, since in this parameter regime α has the most effect. For the lower bound a small positive value ($\alpha < 0.4$) suffices, so the lower bound was chosen to be 0.2.

To summarize the region of parameter space considered for the interpolation in this thesis is $(\alpha, C) \in [0.2, 1.2] \times [0.4, 1.2]$. After constraining the parameter space, the next step is to choose which points in parameter space to simulate and which interpolation method to use.

4.3.2 Choices for simulated parameters

First the choice of simulated points will be discussed. This is a two step process. In the first step use an equidistant grid over $[0.2, 1.2] \times [0.4, 1.2]$. For the distance between grid points choose for both the α and the C direction 0.2, such that the distance between grid points is not too small, which prolongs simulation time; on the other hand the distance is not be too large, such that the broad behavior of λ as a function of α and C can be obtained. The second step makes use of the first step, to improve the interpolation. In this step λ as function of α and C is compared to enclosed numerical particles from the N -body simulation by eye to discuss the evolution behavior as a function of α and C and to further constrain the region, where the optimal parameter pair lies. This is exemplary shown in figure 6, where the enclosed expected number of numerical particle at a radius of 0.1 kpc for all halos, but Halo2sig80 and at a radius of 0.01 kpc for Halo2sig80 is compared to the enclosed number of numerical particles from the N -body simulations. The parameters chosen for the fluid simulation were the corners of the considered region of parameter space, as well as $C = 0.8$ and $\alpha = 0.4, 1$.

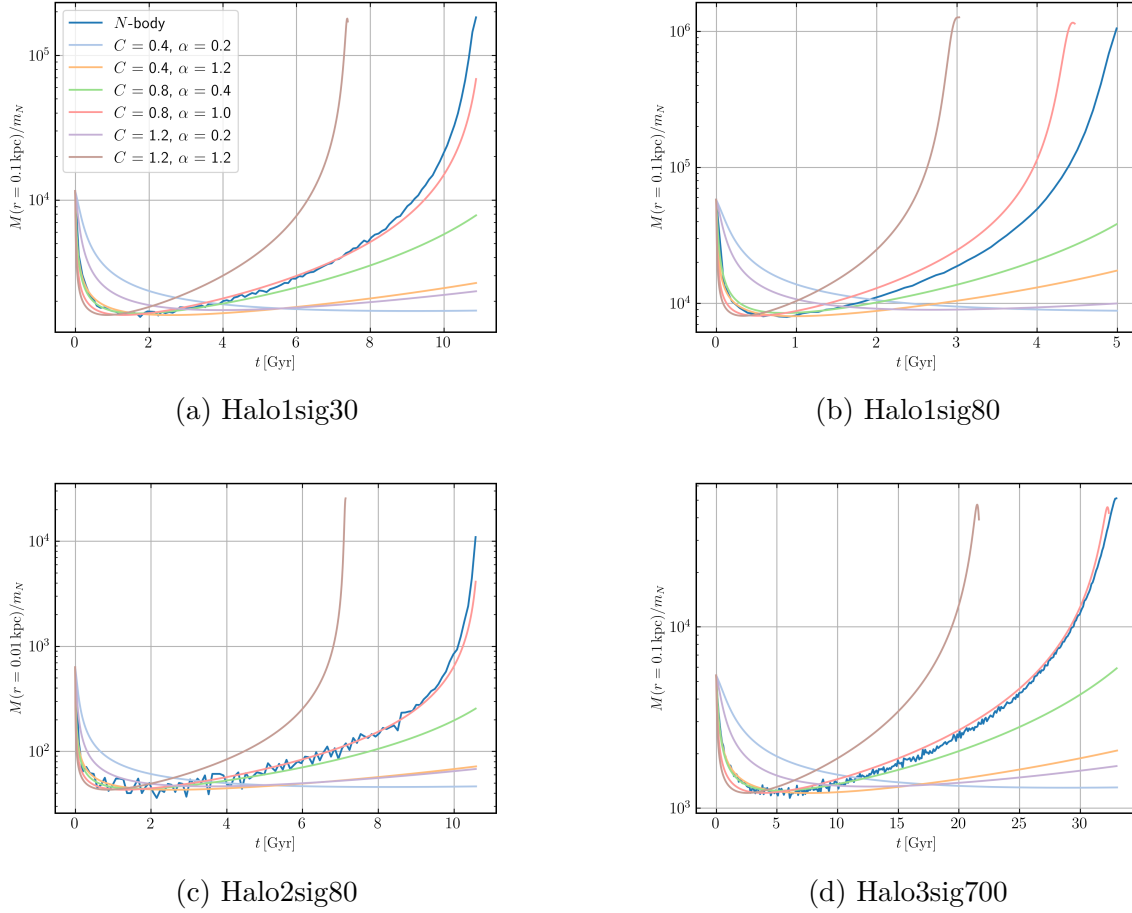


Figure 6: The (expected) number of enclosed numerical particles is shown as function of time. a.) shows it for Halo1sig30 at a radius of 0.1 kpc, b.) shows it for Halo1sig80 at a radius of 0.1 kpc, c.) for Halo2sig80 at a radius of 0.01 kpc and d.) for Halo3sig700 at a radius of 0.1 kpc. The parameters chosen for the fluid simulation were the corners of the considered region of parameter space, as well as $C = 0.8$ and $\alpha = 0.4, 1$.

From figure 6 it can be seen that the main effect of changing the parameters is changing the time the halo spends in the cored phase; increasing α or C results in an earlier collapse time. Further it can be observed for all four halos that the collapse time is too small for $C = \alpha = 1.2$. On the other hand, if $\alpha = 0.2$ the collapse is prolonged by too long to be a viable solution. This appears to be also true for the $C = 0.8, \alpha = 0.4$ case but not so extreme as the $\alpha = 0.2$ cases. On the other hand the $C = 0.8, \alpha = 1$ case seems to roughly get the correct behavior for the enclosed particle numbers, although in the cases of Halo1sig80 and Halo3sig700 λ is slightly larger than N and for Halo1sig30 and Halo2sig80 λ is lower than N . Although the latter part can be fixed, when choosing a slightly larger value for C . At the same time α needs to be adjusted, since a larger value for C would lead to an earlier collapse and increases the discrepancy seen for Halo1sig80 and Halo3sig700. This suggests that the optimal fit has $C \gtrsim 0.8$ and $0.4 \lesssim \alpha \lesssim 1$. To improve the interpolation in this region of parameter space, further simulations were run with $C = 0.85, 0.9, 0.95$ and α between and including 0.4 and 1 in 0.05 steps for each new

value of C .

The simulated points of parameter space are diagrammatically shown in figure 7. The black dots represent the simulation points on the grid, as described in the first paragraph of this section. The red dots represent the simulation choices used to improve the interpolation as described in the second paragraph of this section.

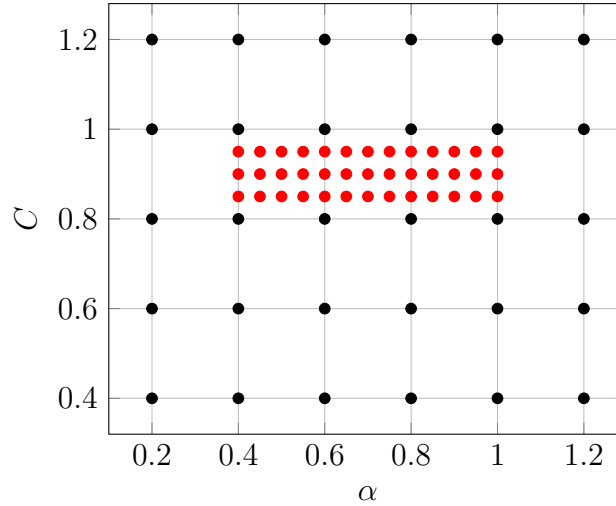


Figure 7: Illustration of the simulated parameters. Each black dot represents a simulated point of the equidistant grid, before any further analysis was conducted. The red marked dots represent the simulation choices used to improve the interpolation in the region, where the optimal fit most likely lies.

4.3.3 Interpolation method

After having simulated all the necessary runs in the fluid approximation an interpolation function $\lambda(\alpha, C)$ needs to be constructed from these simulations for all radii and times. One difficulty arises due to the fact that the simulations are not on a grid. This has two reasons: the first being that the simulations are not on a grid to begin with and the second is that simulations with sufficiently large α and C values collapses earlier than the N -body simulation. This can be seen in figure 6 when looking at the brown curve; for all halos the collapse happens at roughly 2/3 of the total of time the N -body simulation. After the collapse the run is stopped. This implies that for later times not all simulated points are available, so even if the original simulations were performed on a grid, the grid would not persist for all times. So the interpolation method needs to build an interpolating function from an irregular set of points. This problem is closely related to another feature, namely that the $\lambda(\alpha, C)$ needs to be callable on the entire considered parameter space for all times. This means that an interpolation method needs to flexibly switch between interpolation and extrapolation.

The optimal interpolation method, which solves these problems, is radial basis function (RBF) interpolation. A description of RBF interpolation can be found in appendix A. For the implementation of RBF interpolation the `RBFInterpolator` method from `scipy`[47] was used. As a kernel function the thin-plate-spline kernel was used. The reason being that a second fit for the shape parameter at every radius and time is avoided,

since the thin-plate-spline kernel is a scale-invariant kernel[36]. A problem that arose with RBF interpolation was that $\lambda(\alpha, C)$ was negative on some parts of the parameter space, which is unphysical since it would imply that the enclosed halo mass is negative. To avoid this problem the absolute value of the RBF interpolation is considered.

As already stated at the beginning of this section the method used here compares the expected enclosed number of numerical particles with the actual enclosed numerical particles at a certain radius. The method is to construct a probability distribution on parameter space, based on the simulated data, i.e. the posterior. The posterior can be determined using DNS, if the prior and the likelihood are known.

4.3.4 Construction of the prior

The prior is readily obtained. The bounds of the interpolation are used, for the same reasons as illustrated in section 4.3.1, as the bounds of the prior as well. Despite it being previously stated, that it is more likely that for the best fit $\alpha \in [0.4, 1]$ and $C \in (0.8, 1)$, a uniform prior is chosen. This is equivalent to saying that there is no prior knowledge within the bounds. The reason is that this argumentation was purely qualitative, which justifies further interpolation and cannot be translated into how much more likely the further constrained parameter region is compared to the rest. Another advantage of a uniform prior is that the prior transformation, necessary in DNS, is very cheap, speeding the actual run up. To conclude, the prior $\pi(\alpha, C)$ is given by

$$\pi(\alpha, C) = \begin{cases} \frac{5}{4}, & \text{for } \alpha \in [0.2, 1.2] \wedge C \in [0.4, 1.2] \\ 0, & \text{else} \end{cases}, \quad (62)$$

where the factor 5/4 is due to normalization.

4.3.5 Construction of the likelihood

As stated in section 3.3.1 the likelihood contains the information about the simulation data as a function of the parameters α and C . In practice the number of (expected) enclosed numerical particles is only available at discrete radii and times, which are snapshots of the fluid or N -body simulation. In general the saved radii and times differ from the fluid and the N -body simulation. In order to still compare both, one must resort to interpolation. Since the N -body simulations are treated as more accurate than the fluid simulations and fluid simulations need to be calibrated against N -body simulations, a comparison only happens at the radii and times that are part of N -body snapshots, in order to retain exact N -body simulation values. The implication is however that the expected number of numerical particles needs to be interpolated both in times and radii. For both a cubic spline interpolation was used. As already stated in the beginning of section 4.3 the comparison between the enclosed expected number of numerical particles λ and the enclosed number of numerical particles is being done at one radius and over all possible times, which are the times included in the N -body snapshot.

Since the number of enclosed numerical particles is a set of discrete values, the problem is turned into a counting problem at each time snapshot. This motivates the Poisson distribution at each snapshot time to model the likelihood well[42]. Up to an irrelevant normalization constant, the likelihood can be then written as

$$\mathcal{L}(\alpha, C) = \prod_{i_t=1}^{i_{t,\max}} \frac{N_{i_t}^{\lambda_{i_t}(\alpha, C)} \exp(-\lambda_{i_t}(\alpha, C))}{N_{i_t}!}, \quad (63)$$

where i_t denotes the i -th snapshot from a N -body simulation.

4.3.6 Dynamic nested sampling set up

The DNS implementation used for this thesis is the Python package `dynesty`[40], [23]. `dynesty` is built on the nested sampling algorithms developed in [38], [37] and the DNS algorithm established in [19]. Furthermore, bounding methods, developed in [8], [9] were used.

DNS strongly depends on the configuration used. The changes from the standard `dynesty` settings are summarized in the following, i.e. if a setting is not mentioned explicitly then the standard setting of `dynesty 3.0.0` is used.

The number of live points was increased from 500 to 2000. The minimum number of live points that should be used in a run can be approximated as¹⁴ $50 \times (\text{number of dimensions}) \times (\text{number of modes})$ [22]. Since the parameter space is two-dimensional, the dimensions of the problem are two and the expected number of modes can be approximated using a local deterministic maximization algorithm on the likelihood function. At a given radius not more than roughly five modes were found, such that the lower limit of live points is about 500. In order to make sure that the number of live points suffices, it was chosen to be 4 times of this minimum.

Sampling directly from the entire unit cube is very inefficient. So during the run, appropriate bounding distributions are constructed, which encompass the remaining prior volume, from which the live points are sampled from. The bounding distribution chosen are overlapping cubes, called *cubes* in `dynesty` configurations. The advantage of this distribution is that it allows, in low-dimensional spaces, as considered in this thesis, to form emergent structure[40].

In order to be conservative with the prior size, the volume of the bounds is increased with by a factor that can be calculated with bootstrapping. The number of bootstrapping realizations was chosen to be 50 as suggested in [8].

4.3.7 Determine the best fit values

As already explained at the beginning of section 4.3, the posterior can be obtained from the likelihood, equation (63), and the prior, equation (62). The best fit at a certain radius for each parameter is the median of it's posterior distribution. From the posterior also the fit errors for both parameters can be obtained. In principle it is assumed that the errors are antisymmetric, so the lower error is given as the difference of the median and the 16 % quantile and the upper error is given as the difference between the 84 % quantile and the median. This implies, that the errors cover 68 % interval around the median and thus correspond to the standard deviation of the normal distribution[14]. For sake of readability, the formal distinction between lower and upper error will be dropped in all

¹⁴When using bootstrapping[22].

following equations¹⁵. This can in principle be done for all radii, which are available in the N -body snapshots. However not all possible radii are suitable fitting candidates.

Problems arise at small radii, since the continuous $\lambda(\alpha, C)$ is fitted against the discrete number of enclosed numerical particles. If the number of enclosed numerical particles is larger than $\mathcal{O}(1)$ for most times, the enclosed particle numbers appear as quasi-continuous, and can be fitted against a continuous quantity. Another related argument against fitting at too small radii is that the relative error of the enclosed numerical particles is large, often close to 1, while this decreases again for larger radii until it becomes negligible. For this reason the lowest radius considered was 0.01 kpc for Halo1sig30, Halo1sig80 and Halo3sig700 and 10^{-3} kpc for Halo2sig80.

However radii that are too large are also problematic. The relative deviations between the enclosed numerical particles and the expected number of numerical particles are maximally $\mathcal{O}(10^{-2})$ at all times for a wide range of parameters. This indicates that the fluid simulations fit the N -body simulations for a wide range of parameters very well. Whereas the intermediate radii are most sensitive to the choice of parameters. So the largest radius considered is 1 kpc for all halos.

Within the $10^{-2} - 10^{-3}$ kpc to 1 kpc range, the fitting procedure can be done for each radius. The parameter values, which are sufficiently close to each other, i.e. it is clear that they are from the same extrema, which shifts with radius, can be grouped together. Such that the groups are meaningful and are considered for any analysis, they must at least have three members. The final value for each parameter for one halo is given as the arithmetic mean over the parameters in the group with the most members. Analogously the final fitting error for each parameter can be defined as the arithmetic mean over all fitting errors of group with the most members. Furthermore the statistical variation of the parameters within the group provides an additional source of errors. This statistical error of a parameter can be quantified with the standard deviation over all respective parameters of the group. To conclude the parameters can be written as

$$\begin{aligned}\alpha &= \bar{\alpha} + \overline{\Delta\alpha}_{\text{fit}} + \Delta\alpha_{\text{statistics}} \\ C &= \bar{C} + \overline{\Delta C}_{\text{fit}} + \Delta C_{\text{statistics}},\end{aligned}\tag{64}$$

where the bar again denotes the arithmetic mean over a group, $\Delta\alpha_{\text{fit}}$ and ΔC_{fit} are the errors from the fit obtained with DNS for α and C respectively. Finally $\Delta\alpha_{\text{statistics}}$ and $\Delta C_{\text{statistics}}$ denote the standard deviations of α and C respectively over a group.

4.3.8 Fitting with the velocity dispersion

An alternative approach to the one described is to use the three-dimensional velocity dispersion instead of the enclosed number of numerical particles. In this case the a least-squares fit is performed for the parameters α and C . Similar to the previous fitting method, the data from the fluid simulations need to be fitted against the N -body data. This can only occur at positions and times, which are part of the snapshots of the N -body simulations, i.e. the velocity dispersion from the N -body simulations does not need to be interpolated. Again the fluid data is known at different times and radii; however, since the velocity dispersion is an intensive quantity, it cannot simply be interpolated in radius

¹⁵Further motivation for this abuse of notation can be found in section 5.4.

and time. Instead the velocity dispersion needs to be re-binned, such that they match the N -body bins. Note that when combining several bins from the fluid simulations, the resulting velocity dispersion has to be weighted by the mass of the respective bins. After re-binning, the velocity dispersion is only known at discrete points in parameter space; to obtain the velocity dispersion at an arbitrary point in parameter space, the velocity dispersion needs to be interpolated at each bin. The RBF interpolation method is used again. In order to be consistent with the fit using the number of enclosed numerical particles, only radial bins between 0.01 kpc and 1 kpc for every halo but Halo2sig80 and 10^{-3} kpc to 1 kpc for Halo2sig80, are used. To summarize, the fitting condition is

$$\sum_{t=0}^{t_{\max}} \sum_{r=r_{\min}}^{1 \text{ kpc}} [\nu(\alpha, C, t, r) - \nu_N(t, r)]^2 \longrightarrow \text{minimize}, \quad (65)$$

where $\nu(\alpha, C, t, r)$ is the binned velocity dispersion of the fluid simulations, $\nu_N(t, r)$ is the velocity dispersion of the N -body simulations and $r_{\min}$ is minimal considered radius. The boundaries for the minimization algorithm are the same as used for the prior in the other fitting method. An advantage of this method is that the fit is not done for each radius individually, but can be done for all relevant radii at the same time. The decisive disadvantage is that not the full likelihood is accessible and thus the posterior can not be constructed. From there neither the errors nor the goodness of the fit can be evaluated[35], making the method of comparing the number of enclosed particles superior; however, this method can still be used as a consistency check.

5 Results and discussion

5.1 Results for fitting the enclosed numerical particles

5.1.1 Results for Halo1sig30

The best fit values for the parameters α and C obtained for Halo1sig30 at each radius are shown in figure 8. However it was stated in section 4.3.7 that only radii between 0.01 kpc and 1 kpc are considered. So only for radii between 0.01 kpc and 1 kpc, the best fit values have been plotted in figure 8.

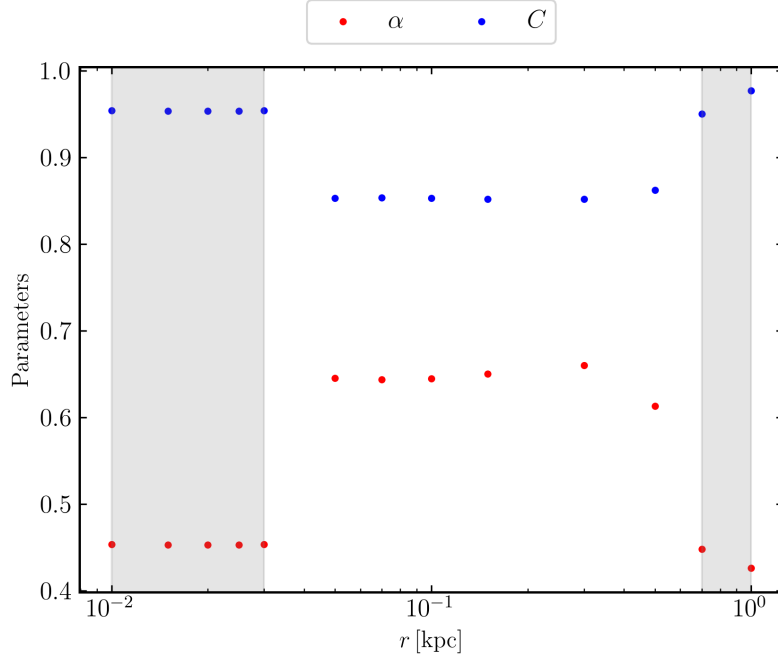


Figure 8: Best fit values are shown for α and C for Halo1sig30 in dependence of the radii, for radii between 0.01 kpc and 1 kpc. Within the gray colored region the parameters are used to construct the halo best fit.

From figure 8 it can be seen that there are two distinct groups of best fit values. The one at $(\alpha, C) \approx (0.45, 0.95)$ extends from 0.01 kpc to 0.03 kpc and also includes points at 0.7 kpc and 1 kpc and counts seven members. The other is at $(\alpha, C) \approx (0.6, 0.85)$ and is located between 0.03 and 0.7 kpc and counts only six members, such that only the first group is considered for the final fit. The parameters used to construct the halo best fit are shown in the gray colored region in figure 8. Taking the arithmetic mean of both the parameters and errors of the group yields

$$\begin{aligned} \alpha_{\text{Halo1sig30}} &= 0.4485 \pm 0.0095 \\ C_{\text{Halo1sig30}} &= 0.9568 \pm 0.0086 \end{aligned} \quad (66)$$

From figure 8 can be seen, that the fit values at $r = 0.7$ kpc, 1 kpc, especially the one at 1 kpc, deviate slightly from the ones between 0.01 kpc to 0.03 kpc, which results in an overall larger mean value for C and lower of α , as well as increasing the overall statistical error.

5.1.2 Results for Halo1sig80

Figure 9 shows the best fit values for the parameters α and C in dependence of the radii. In the case of Halo1sig80 only the radii between are 0.01 kpc and 1 kpc are considered as well. This means, that only for radii between 0.01 kpc and 1 kpc, the best fit values have been plotted.

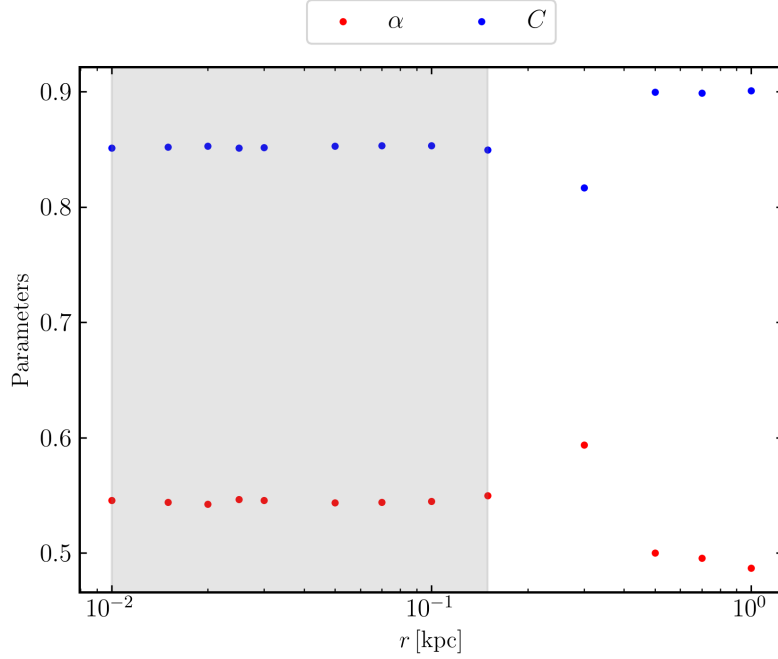


Figure 9: Best fit values are shown for α and C for Halo1sig80 in dependence of the radii, for radii between 0.01 kpc and 1 kpc. Within the gray colored region the parameters are used to construct the halo best fit.

From figure 9 it is evident that the most dominant group clearly is the one at $(\alpha, C) \approx (0.55, 0.85)$, which extends from 0.01 kpc to 0.15 kpc. So this group is used for the final fit. Despite not being as dominant as for Halo1sig30, a group at $(\alpha, C) \approx (0.45, 0.95)$ with 3 members, extending from 0.5 kpc to 1 kpc, is still present. This group is shown in the gray colored region in figure 9. Taking the arithmetic mean of both of the parameters and errors of the former group yields

$$\begin{aligned} \alpha_{\text{Halo1sig80}} &= 0.5453 \pm 0.0021 \\ C_{\text{Halo1sig80}} &= 0.852 \pm 0.0011 \end{aligned} \quad (67)$$

Compared to Halo1sig30 the scatter in the data is larger; however the statistical error does not necessarily increase the statistical error, since the statistical error in the case of Halo1sig30 was already drastically increased due to *outliers*.

5.1.3 Results for Halo2sig80

The best fit values for the parameters α and C can be obtained for Halo2sig80 at each radius and are shown in figure 10. Contrary to the previous two halos, radii between 10^{-3} kpc and 1 kpc are considered. So only for radii between 10^{-3} kpc and 1 kpc, the best fit values have been plotted in figure 10.

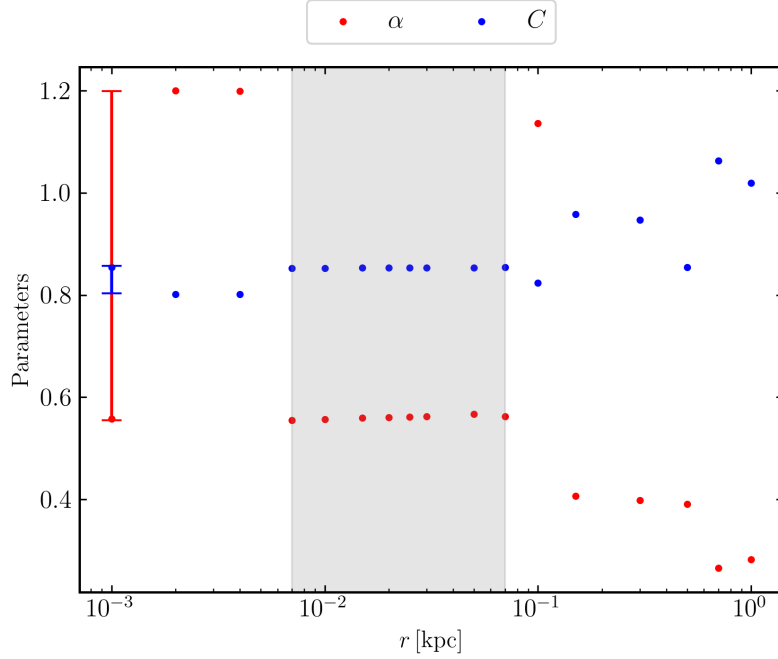


Figure 10: Best fit values are shown for α and C for Halo2sig80 in dependence of the radii, for radii between 10^{-3} kpc and 1 kpc. Within the gray colored region the parameters are used to construct the halo best fit.

From figure 10 it can be seen that the dominant group is the one at $(\alpha, C) \approx (0.55, 0.85)$, which extends from $7 \cdot 10^{-3}$ kpc to 0.07 kpc. Contrary to Halo1sig30 and Halo1sig80, which appeared to have a clear group at $(\alpha, C) \approx (0.45, 0.95)$, Halo2sig80 appears not to have such a group, leaving the group at $(\alpha, C) \approx (0.55, 0.85)$ as the only one usable for the fit. Another feature, which is different from Halo1sig30 and Halo1sig80 is that for a radius, which can be considered for the halo best fit, $r = 10^{-3}$ kpc, errors for α and C are far larger compared to the errors at other radii. For that reason error bars are shown in figure 10 for $r = 10^{-3}$ kpc. The origin of the large errors is that there is another maximum of the posterior at $(\alpha, C) \approx (0.8, 1.2)$, which is also present at larger radii but less dominant in terms of posterior mass. This maximum is also most likely spurious, since the maximum is at the boundary of the interpolation and the interpolation can not capture the behavior of the λ with $\alpha > 1.2$ well, as explained in section 4.3.1. The less dominant posterior mass implies, that this maximum is outside of one standard deviation, thus it is not visible with the errors. The only exceptions being $r = 0.002, 0.004$ kpc, where this maximum actually dominates the posterior, meaning it has the most posterior mass, at these two radii. The consequence is that due to the large errors the best fit parameters at $r = 10^{-3}$ kpc are not considered for the halo fit. Error bars for other radii are not shown, since they are too small to be properly noticeable. This also holds true for the other halos and thus also for the other halos no error bars are shown. The consequence of that is further discussed in section 5.5. Taking the arithmetic mean of both the parameters and errors of the group at $(\alpha, C) \approx (0.55, 0.85)$ group yields

$$\begin{aligned} \alpha_{\text{Halo2sig80}} &= 0.5609 \pm 0.0041 \\ C_{\text{Halo2sig80}} &= 0.8532 \pm 0.0008 \end{aligned} \quad (68)$$

In contrary to Halo1sig30 and Halo1sig80, the best fit values at the individual radii are closer together. This holds especially for C , where the deviation from the mean is about an order of magnitude less. This results in the very small statistical error for C .

5.1.4 Results for Halo3sig700

Figure 11 shows the best fit values for the parameters α and C as a function of radius. Analogous to the first two halos, radii between 0.01 kpc and 1 kpc are considered. So only the best fit values for radii between 0.01 kpc and 1 kpc are shown in figure 11.

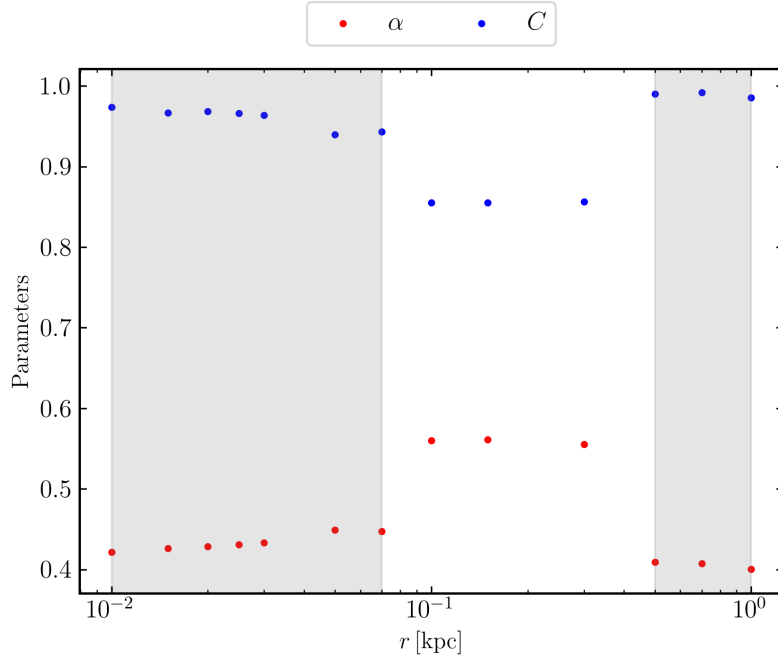


Figure 11: Best fit values are shown for α and C for Halo3sig700 in dependence of the radii, for radii between 0.01 kpc and 1 kpc. Within the gray colored region the parameters are used to construct the halo best fit.

From figure 11 it can be observed that the dominant group is the one at $(\alpha, C) \approx (0.45, 0.95)$, which includes the parameters at every radius, except for $r = 0.1, 0.15, 0.3$ kpc. This implies that the group at $(\alpha, C) \approx (0.45, 0.95)$ is the one used for the fit. However, similar to Halo1sig30, a second group with three members exists at $(\alpha, C) \approx (0.55, 0.85)$, with corresponding radii $r = 0.1, 0.15, 0.3$ kpc. Taking the arithmetic mean of both the parameters and errors of the group yields

$$\begin{aligned} \alpha_{\text{Halo3sig700}} &= 0.4254 \pm 0.0157 \\ C_{\text{Halo3sig700}} &= 0.9689 \pm 0.017 \end{aligned} \quad (69)$$

Compared to the other halos, the scatter in both α and C is fairly large, containing multiple points that deviate about 0.02 from the respective mean value. This results in greater statistical errors, which explains why the errors for the fit of Halo3sig700 are at least an order of magnitude larger than the errors for the other fits.

5.2 Construction of the universal fit

A natural question, which arises is the one regarding the existence of a parameter pair, which describes the evolution of a wide range of halos very well. A remarkable feature is that for all halos analyzed, many fit values appear to group around $(\alpha, C) \approx (0.45, 0.95)$, $(\alpha, C) \approx (0.55, 0.85)$ and $(\alpha, C) \approx (0.6, 0.85)$. With the first one appearing for three out of the four halos, while the second appears in all halos, but Halo1sig30, and is the group used for the best fit for Halo1sig80 and Halo2sig80. The last one only appears in the case of Halo1sig30. However the groups at $(\alpha, C) \approx (0.55, 0.85)$ and $(\alpha, C) \approx (0.6, 0.85)$ match the values for C and the value for α only differs by 0.05; both groups should lead to similar halo evolutions and thus the group at $(\alpha, C) \approx (0.6, 0.85)$ is absorbed into the one at $(\alpha, C) \approx (0.55, 0.85)$. This is further compensated by a higher uncertainty for value for α of the universal fit, due to a larger statistical error. In the case of Halo1sig30 the group has only one member less than the group at $(\alpha, C) \approx (0.45, 0.95)$ and is thus comparable in size. Whereas in the case of Halo3sig700 the group is with three members still present, but subdominant compared to the group at $(\alpha, C) \approx (0.45, 0.95)$.

This suggests that all halos have best fit values, at least for some of the radii, at $(\alpha, C) \approx (0.55, 0.85)$. The same fitting procedure, which has been applied to obtain the best fit value for a halo can also be applied Halo1sig30 and Halo3sig700 for the $(\alpha, C) \approx (0.55, 0.85)$ group to obtain *best fit* values at $(\alpha, C) \approx (0.55, 0.85)$. Such that for every halo a fit, with additional errors is obtained at $(\alpha, C) \approx (0.55, 0.85)$. With these four values per parameter, the universal fit of a parameter is constructed by taking the arithmetic mean of these four values. Furthermore the errors of the individual fits can be taken into account by taking the arithmetic mean of all the errors. The standard deviation of the four values per parameter is used to take the statistical error into account.

$$\begin{aligned}\alpha_{\text{univ}} &= \bar{\alpha}_{\text{halos}} + \overline{\Delta\alpha}_{\text{halos}} + \Delta\alpha_{\text{statistics}} \\ C_{\text{univ}} &= \bar{C}_{\text{halos}} + \overline{\Delta C}_{\text{halos}} + \Delta C_{\text{statistics}},\end{aligned}\tag{70}$$

where α_{univ} , C_{univ} represent the universal fits, $\bar{\alpha}_{\text{halos}}$, $\bar{C}_{\text{halos}}$ are the mean values of the halo best fits for α and C . $\overline{\Delta\alpha}_{\text{halos}}$ and $\overline{\Delta C}_{\text{halos}}$ are the mean errors respectively and $\Delta\alpha_{\text{statistics}}$, $\Delta C_{\text{statistics}}$ are the respective statistical errors. Inserting values into equation (70) yields

$$\boxed{\begin{aligned}\alpha_{\text{univ}} &= 0.57 \pm 0.027 \\ C_{\text{univ}} &= 0.857 \pm 0.007\end{aligned}}.\tag{71}$$

5.3 Results of the velocity dispersion fit

The results for the best fit values obtained by fitting the velocity dispersion are shown in table 4.

Table 4: Table denoting the best fit values for the parameters α and C obtained by fitting the velocity dispersion. The values in the brackets are values obtained from local minima better coinciding with the method of fitting the number of enclosed numerical particles. The values shown are rounded to the second significant digit.

Name	Halo1sig30	Halo1sig80	Halo2sig80	Halo3sig700
α	0.46	0.5 (0.54)	1.2 (0.56)	0.44
C	0.96	0.91 (0.86)	0.85 (0.87)	0.96

From table 4 it can be seen that the obtained fit values for Halo1sig30 and Halo3sig700 are barely within the margins of errors of the fitting values obtained with from the other fitting method. On the other hand, the α value for Halo2sig80 is far above the best fit value determined from fitting the number of enclosed numerical particles. However in section 5.1.3 it was noticed that there exists a maximum of the posterior for $(\alpha, C) \approx (1.2, 0.8)$, which is subdominant for most considered radii but is still present, i.e. it has less associated posterior mass than the one at $(\alpha, C) \approx (0.55, 0.85)$. Despite the observed difference in C of ≈ 0.05 , both extrema most likely have the same origin. The reason is, that it was noted in section 5.1.3, that the extremum is most likely spurious, due to interpolation difficulties. These difficulties could also explain the noticed difference in C . When fitting the velocity dispersion there is also a local minimum given, which is close to the one obtained by the enclosed numerical particles fit. The parameters for the local minimum are given in the brackets in table 4. The value for α is within the margin of error of the value for α obtained with from the previous fitting method, while the value for C differs by roughly 0.02 and hence is outside the margin of error. The reason for that can be errors in the interpolation, which are unaccounted in the analysis. A similar point can be made for Halo1sig80. In this case the fit values obtained from fitting the velocity dispersion differ by ≈ 0.05 from the ones obtained by fitting the number of enclosed numerical particles. However there is also a local minimum, which coincides well with the best fit values obtained from fitting the number of enclosed numerical particles. The corresponding parameters are given in brackets in table 4. These parameters are barely outside the margin of error for the fit values obtained by fitting the number of enclosed numerical particles. These deviations can be explained again by interpolation errors. On the other hand in the case the fit with the number of enclosed numerical particles, the posterior has a local maximum, which has too little posterior mass to be properly noticed, which coincides with the best fit values obtained by fitting the velocity dispersion well. A local extremum means, that the fit values are locally optimal[7] and since both methods share local extrema, they also share good fit values. A more physical reason why both are good fit values can be seen from the fact, that larger values of α and C lead to an earlier collapse. Since the α value of the velocity dispersion fit is about 0.05 smaller than the value of the enclosed numerical particles fit and the respective C value 0.05 larger, changes in the collapse time are small. As can be seen from section 5.5 getting the collapse time correct is important for the accuracy of the fit. As explained in section 3.3.1, DNS does not find the global extremum, contrary to a least squares approach[35]. Thus both methods are not expected to yield the same result and cannot be further compared.

To conclude, both methods can be difficult to compare, since an extremal value is compared

to the posterior mass; however, both approaches suggest consistency.

5.4 Structure of the fits

In order to illustrate the structure of the fits at the individual radii, the posterior distributions for all four halos are shown in figure 12. The dark blue region represents the 39.3 % confidence region, the blue region the 86.4 % confidence region and the lightblue region the 98.9 % confidence region. The best fit for α and C are shown by semi-transparent black lines, both in the histograms and in the posterior. Above the histograms the best fit values are shown with the respective errors. They are calculated just as explained in section 4.3.7.

The fit is shown at $r = 0.01$ kpc, although their structure holds almost universally. Notable exceptions are some radii, which were excluded to begin with, since the number of numerical particles is not quasi-continuous and thus a fit is unpractical, as explained in section 4.3.7 and the $r = 10^{-3}$ kpc fit for Halo2sig80, where there are two dominant maxima. So it can be concluded that there exists a unique best fit at a given radius, especially for the radii used to construct the halo best fit and universal fit.

From all histograms in figure 12 it becomes clear, that the distributions are not perfectly symmetric Gaussian's. This shows that the errors have a slight asymmetry, as was already stated in section 4.3.7. However it can also be observed that this asymmetry is sufficiently small such that it does not have an effect within four decimal digits, at least at the radii used to construct the halo best fit. The implication is that the errors for the halo best fit values and the universal fit appear symmetric. This also justifies the abuse of notation in equation (64), where the asymmetry was formally ignored. Any cases with noticeable antisymmetric errors, for example as shown in figure 10, coincide with radii mentioned in previous paragraph at which the structure of the fit shown in figure 12 does not hold up.

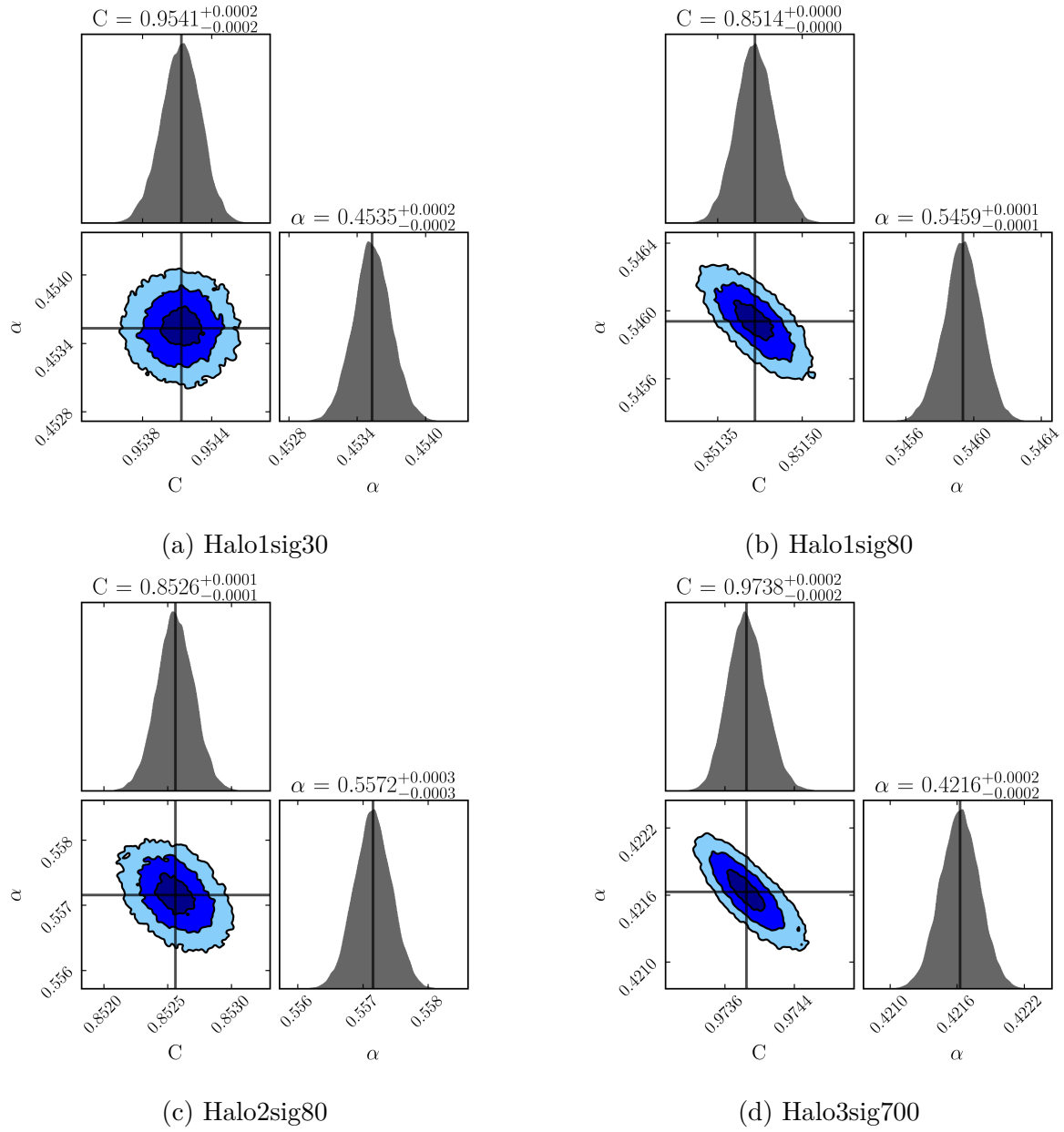


Figure 12: Corner plots showing the posterior distribution for all four halos at $r = 0.01$ kpc. The dark blue region represents the 39.3 % confidence region, the blue region the 86.4 % confidence region and the lightblue region the 98.9 % confidence region. Above the histograms the respective best fit values with errors are shown. The semi-transparent black lines in the histogram represent indicate the best fit values for α and C respectively.

5.5 Discussion of the fit values

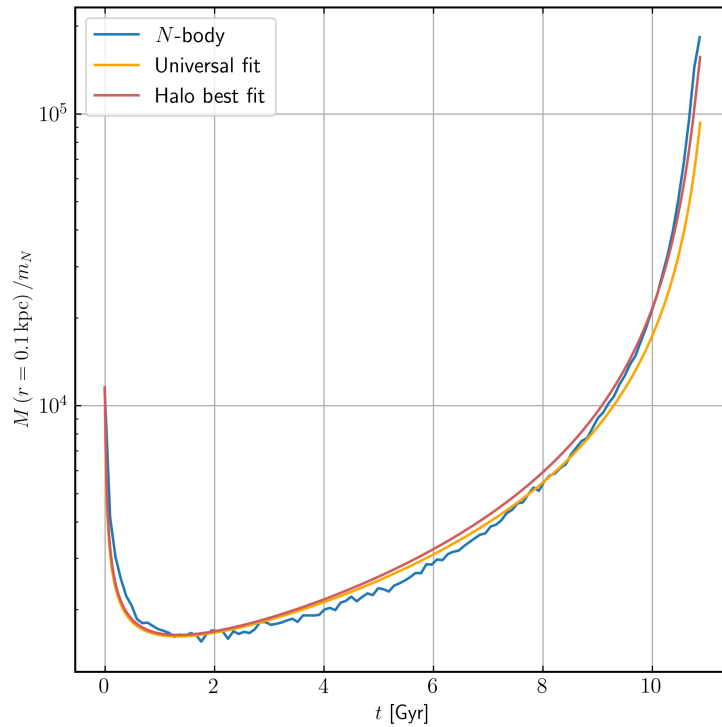


Figure 13: The (expected) number of enclosed numerical particles for Halo1sig30 at a radius of 0.1 kpc is shown as function of time. The parameter pairs chosen are the universal fit in orange and the halo best fit in red.

As can be seen from figure 13 the collapse is delayed slightly for the halo best fit and especially for the universal fit, such that they both underestimate the number of enclosed numerical particles during the collapse phase and overestimate the number of numerical particles in the cored phase. On the other hand the halo best fit overestimates the enclosed number of numerical particles more than the universal fit at the cored phase. However the collapse phase has for smaller radii up to several orders of magnitude of numerical particles; hence a more correct modelling of the collapse phase is statistically more relevant than the cored phase. The underestimation of the number of expected numerical particles during the collapse phase is due to a delay in the collapse. In the case of the universal fit, the collapse is delayed by ≈ 200 Myrs, while the halo best fit is only delayed by ≈ 70 Myrs. The collapse time is approximated from figure 13 by determining the difference between the final time shown and the time, where the number of enclosed numerical particles best matches the number of expected numerical particles at the final time. This analogously done for the other halos as well.

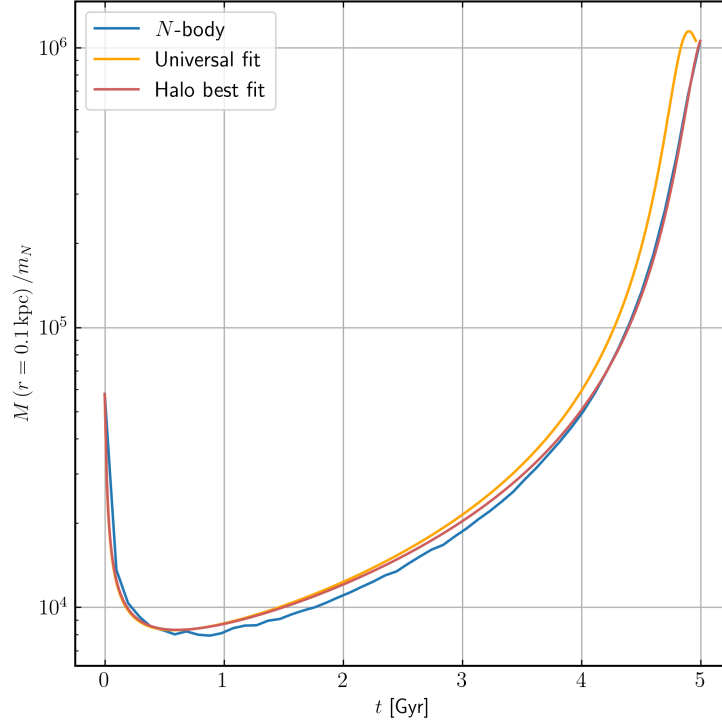


Figure 14: The (expected) number of enclosed numerical particles for Halo1sig80 at a radius of 0.1 kpc is shown as function of time. The parameter pairs chosen are the universal fit in orange and the halo best fit in red.

From figure 14 it can be observed that the halo best fit fits the N -body data almost perfectly. Contrary to Halo1sig30, the halo best fit is also better in the cored phase than the universal fit, since the collapse is accelerated by ≈ 150 Myrs for the universal fit.

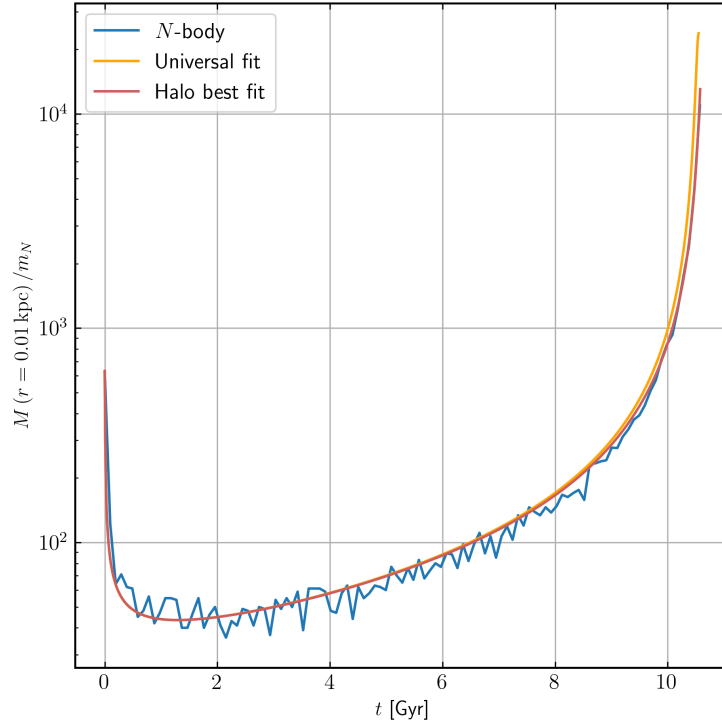


Figure 15: The (expected) number of enclosed numerical particles for Halo2sig80 at a radius of 0.01 kpc is shown as function of time. The parameter pairs chosen are the universal fit in orange and the halo best fit in red.

It can be observed from figure 15 that the halo best fit almost fits the N -body data perfectly, while on the other hand the universal fit again similar to Halo1sig80 accelerates the collapse by ≈ 90 Myrs. This results in an over estimate of the number of enclosed numerical particles during the collapse phase. This effect is especially visible at smaller radii, where the number of enclosed numerical particles varies by several orders of magnitude during collapse and hence small variations in the collapse time can lead to visibly different results.

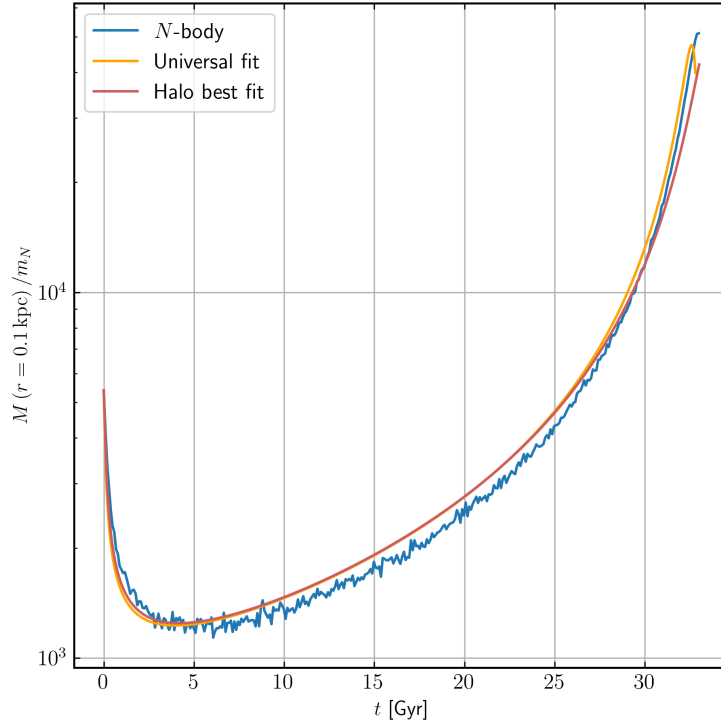


Figure 16: The (expected) number of enclosed numerical particles for Halo3sig700 at a radius of 0.1 kpc is shown as function of time. The parameter pairs chosen are the universal fit in orange and the halo best fit in red.

In the case of Halo3sig700 it can be observed in figure 16 that the halo best fit delays the collapse by ≈ 460 Myrs, while on the other hand the collapse is accelerated for the universal fit by ≈ 420 Myrs. Although this deviation is an approximately four times above the other halos, since the total time of the halo evolution is roughly 3-6 times the time of the evolution of the other halos, the deviation relative to the total halo time should be comparable. Although it appears, that the universal fit more accurately describes of collapse time, the difference between collapse time of the universal fit and the halo best fit is only ≈ 40 Myrs. This is negligibly short, compared to the total halo evolution.

It needs to be remarked that, although the number of enclosed (expected) numerical particles over time is shown for only one radius explicitly, the qualitative conclusions drawn above hold up for at least the range of radii, which is considered for the fit in the first place. The behavior, that the properties of the halo of the fluid simulations do not change much with radius, allows for the formulation of a halo best fit. This means, that the halo best fit also describes the evolution well at radii, which were not used to construct the halo best fit. It further needs to be noted that the construction of the halo best fit does not involve fitting the best fit values of the parameters at the individual radii to a parameter, which is constant as a function of radius. As a matter of fact α and C are not viewed as functions of radius, since there is no known functional behavior to begin with and from the analysis conducted in section 5.1 no universal empirical law for either α or C seems to exist. Since there is no known behavior of the parameters as a function of radius, the mean value of the largest group of parameters is further justified as the halo best fit. The mean value further provides a fit value at radii, which are not part of the N -body snapshots and thus cannot be fitted individually. Furthermore, the mean value also makes the fit efficient

to use in practice, since only one simulation is required to calculate the enclosed number of numerical particles at two distinct radii and the best fit values easily translatable to other observables, like the velocity dispersion. However, this implies, that the halo best fit is most likely not in general statistically optimal at every radius, but a good compromise for all radii. The fact that this is not statistically optimal at every radius can be seen in figures 8, 9, 10 and 11, since the errors of the individual fits are so small, that they are not even shown. Meanwhile statistical variations are sufficiently large to be noticeable. The consequence is, that the best fit value of the group is rarely within the margin of error at an individual radius. Note that, it is not expected that the mean value of a parameter is within the margin of errors of the respective parameters at the individual radii, since the fits at different radii are completely separate and independent from each other. The fact that this compromise between radii works is further motivated by the fact that these statistical variations are sufficiently small such that the halo evolution still closely matches the N -body simulation, as can be seen in figures 13, 14, 15 and 16.

Another evident aspect is that fits with smaller errors, as calculated in section 5.1, perform visibly better than fits with larger errors. This is also expected, since sufficiently large errors in either parameter are due to the statistical error dominating over the fitting error. This leads to larger deviations of the mean from the statistical best fit values at the individual radii and thus to larger deviations in the number of enclosed numerical particles.

From the figures in this section, as well it is also evident, that the gravothermal fluid model tends to overestimate the expected number of numerical particles during the cored phase. This leads to relative deviations of up to approximately 10 %. The only seeming exception being Halo2sig80; however when repeating the analysis for a radius of 0.05 kpc, it also becomes apparent, that the number of expected enclosed numerical particles is also overestimated. Another deviation can be seen in figure 16, where the the number of expected enclosed particles is slightly underestimated during the late stages of the collapse phase. This effect is not observed for the other halos and thus appears to be halo specific. These systematic deviations most likely stem from the simplifying assumptions made by the fluid model, compared to the N -body simulation.

A source of uncertainty that influenced the fits is the error made by the interpolation, when constructing the interpolation function $\lambda(\alpha, C)$. Since the method of fitting the number of enclosed numerical particles is consistent with the method of fitting the velocity dispersion and the interpolating function differs, the interpolation error can at most be ≈ 0.01 . This corresponds to the approximate maximal difference between the halo best fit from, when fitting the number of enclosed numerical particles, with the respective minimum from the velocity dispersion fit.

The universal fit has the largest errors and thus the halo best fits perform visibly better than the universal fit; however, the universal fit still gets for all four halos the halo evolution roughly correct, just the collapse has been delayed or accelerated by less than 10 % of the total halo evolution. The collapse is delayed for Halo1sig30, because in the construction of the universal fit especially the parameter α is underestimated for Halo1sig30, since the other α values used to construct the universal fit are lower than the one for Halo1sig30. Meaning that the value of α for the universal fit is also lower and thus delays the collapse. On the other hand the collapse is accelerated slightly for the other halos. The reason for this is complementary to the one why the collapse is delayed for Halo1sig30. The

universal fit value for α get's increased since the α value for Halo1sig30 used to construct the universal fit is slightly larger than the value for α of the other halos, thus accelerating the collapse. Overall this means that for a halo, which has not been explicitly fitted to a N -body simulation, the universal fit will provide a fairly accurate description of the halo evolution and is usable, if the halo best fit is not known. As can be noticed from table 2 the four halos considered cover a wide range of cross sections, masses and concentrations. This implies that the calculated universal fit parameters are quasi-independent of mass, concentration and of cross section. The fact that the α value for Halo1sig30 used to construct the universal fit differs from the other ones by ≈ 0.05 can be due to some large statistical variation, since the other α values also scatter by ≈ 0.01 , so in the same order of magnitude, and there is no other indication of a dependence of α on cross section. As long as there is no other compelling evidence of parameter dependence on cross section, the reported difference in α is assumed to be due to a statistical variation.

As already stated in section 4.3.1 literature regarding fit values for α and C to which the fit values can be compared to is sparse. One such study[26] suggests that $\alpha = 0.899$ and $C = 0.949$ fit the N -body data well, independent of halo mass, concentration or cross section. This is similar to the universal fit values found in this thesis, which are also independent of halo mass, concentration or cross section. However the the universal fit values found in this thesis¹⁶ differ too much for these two approaches to be consistent. The values for C obtained in [26] more closely matches the halo best fit values for C for Halo1sig30 and Halo3sig700. However the value for α obtained in [26] is approximately twice the best fit value for α for Halo1sig30 and Halo3sig700, indicating the fit parameters found in [26] would result in a collapse that is far too early. The difference between these two different obtained fit values most likely stems from differences in the N -body simulations used to fit the gravothermal fluid model against. The authors of [26] use the N -body code AREPO with converge guidelines developed in [27], while in this thesis OpenGadget 3 was used for the N -body simulations, which can differ, for example, in simulation methods, parameters or boundary conditions[26]. The inconsistency between different N -body simulations was also noticed in [26]. Another study[45] suggests that $(\alpha, C) = (0.512, 0.885)$ and $(\alpha, C) = (0.6, 0.8)$ describe the halo evolution well. These values are relatively consistent with the universal fit values obtained in this thesis. Deviations can be mainly traced back to methodological differences between [45] and this thesis. The *early mass transfer*, which was noted in [45], is just a numerical artifact of the performed analysis.

6 Conclusion and outlook

In this thesis the two free parameters α and C in the gravothermal fluid model were determined by comparing the model with N -body simulations. In total four halos with varying cross section, mass and concentration were studied. To obtain α and C for each halo, the number of enclosed expected enclosed numerical particles, defined as the enclosed mass divided by the mass of an N -body simulation particle, was fitted against the number of enclosed numerical particles at each radius. From the best fit values at each radius a best fit for the entire halo can be constructed. As a consistency check, best fit values for the parameters were also obtained by fitting the velocity dispersion from the gravothermal fluid simulations against the velocity dispersion of the N -body simulations. Overall consistency was observed between these two methods. Furthermore, a universal fit was also constructed.

¹⁶ $\alpha_{\text{univ}} = 0.57, C_{\text{univ}} = 0.857$

The universal fit is quasi-independent of mass, concentration and of cross section. Here the reported scatter in α is seen as uncorrelated to the halo cross section. In general however the halo-specific best fit values appear to be more accurate and have the smaller uncertainties than the universal fit. However the universal fit still gives a fairly accurate description of the halo evolution and thus can be used if no halo-specific fit for α and C is available.

For the next steps, one can translate errors in the parameters $\Delta\alpha$ and ΔC into errors of the physical observables, primarily the enclosed mass. Furthermore, it was mentioned that different N -body simulations lead to different results; one can further study this by determining the parameters α and C with other N -body simulations. In this thesis pure SIDM halos were studied, neglecting other effects such as baryons. In the future, also baryonic effects on the parameters α and C can be studied. In section 4.3.1 it was shown that changes in the parameter α have a diminishing effect, if $\alpha \gtrsim 1$ and thus provides a natural boundary for the interpolation and the DNS runs. This effect can be further studied by considering larger α values than were studied in this thesis. Furthermore, it was noted that the α value for Halo1sig30 used to construct the universal fit deviates by ≈ 0.05 from the other α values. This was interpreted as uncorrelated to the halo parameters, especially since no other dependence of α on the halo parameters was noted. This can be further studied, when more halos with cross sections comparable to the one of Halo1sig30 are calibrated. Even if there is no statistical correlation between cross section and parameters, further studied halos could help with a better treatment of outliers.

A Radial basis function interpolation

The principle assumption behind RBF interpolation is that every known point isotropically affects its surroundings. This *effect* is quantified by the kernel function $\phi = \phi(r)$, where r denotes the Euclidean distance between the known point and the point where the interpolation function is calculated. Consider that a function y is known at discrete points $y(\mathbf{x}_i)$ at $\mathbf{x}_i$ with $i = 1, \dots, N$. Hence the interpolating function $y(\mathbf{x})$ at a position $\mathbf{x}$ can be written as a linear combination of kernel functions[35]

$$y(\mathbf{x}) = \sum_{i=1}^N w_i \phi(|\mathbf{x} - \mathbf{x}_i|), \quad (72)$$

where the w_i 's are weights. The weights are determined by making use of the known function values $y(\mathbf{x}_j)$ with $j = 1, \dots, N$. With these points a $N \times N$ system of linear equations can be constructed, which needs to be solved for the weights w_i [35]

$$y(\mathbf{x}_j) = \sum_{i=1}^N w_i \phi(|\mathbf{x}_j - \mathbf{x}_i|) \quad \text{for } j = 1, \dots, N. \quad (73)$$

There are various kernel functions in literature; some of these can be found in table 5. Nearly all kernel functions are dependent on a scale factor r_* ; exceptions are the so-called *scale-invariant* kernels, like the thin-plate-spline [36]. For non-scale-invariant kernels, getting the correct r_* is critical, since r_* heavily influences the accuracy of the interpolation[35].

Table 5: Table denoting some famous kernel functions. Functions taken from [35].

Kernel	$\phi(r)$
multiquardic	$\sqrt{r^2 + r_*^2}$
inverse multiquardic	$\frac{1}{\sqrt{r^2 + r_*^2}}$
Gaussian	$\exp\left(-\frac{r^2}{2r_*^2}\right)$
thin-plate-spline	$r^2 \ln\left(\frac{r}{r_*}\right)$

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Ehrenwörtliche Erklärung/ Statutory Declaration

An das Ende von schriftlichen Arbeiten (bspw. Seminararbeiten, Bachelor- oder Masterarbeiten) ist eine ehrenwörtliche Erklärung zu setzen. Sie hat folgenden Wortlaut:

Erklärung nach § 39 (15) Prüfungsordnung 2020 für den BAMA-Studiengang Physik

Hiermit erkläre ich, dass ich die Arbeit selbstständig und ohne Benutzung anderer als der angegebenen Quellen und Hilfsmittel verfasst habe. Alle Stellen der Arbeit, die wörtlich oder sinngemäß aus Veröffentlichungen oder aus anderen fremden Texten entnommen wurden, sind von mir als solche kenntlich gemacht worden. Ferner erkläre ich, dass die Arbeit nicht - auch nicht auszugsweise - für eine andere Prüfung verwendet wurde.

Ort, Datum: Unterschrift: