
On the Applicability of Gravothermal Fluid
Approaches for Velocity-Dependent SIDM:
A Direct Comparison with N-body Simulations

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Abstract

Self-Interacting Dark Matter (SIDM) research proposes velocity-dependent cross-sections to explain a wide variety of astrophysical observations. This work focuses on the analysis of the integrations of the latter in gravothermal fluid models. To this end, various approaches from the literature that share the same physical basis but differ in their implementation are analyzed. This serves to validate the author's own results as plausible through comparative argumentation. This step is particularly necessary because significant differences were found for all literature models examined when compared directly with N -body data. The methodological basis of this work thoroughly introduces the models and their differences as well as the resulting implications. In terms of accuracy, the analysis leaves one central conclusion open: Either there is a systematic error in the underlying dataset, which is considered extremely unlikely, or the approach established in the literature for coupling phase space dynamics and fluid models is oversimplified.

Zusammenfassung

Die Selbstinteragierende-Dunkle-Materie-Forschung schlägt geschwindigkeitsabhängige Wirkungsquerschnitte vor, um verschiedenste astrophysikalische Beobachtungen zu erklären. Diese Arbeit beschäftigt sich mit der Analyse von Integrationen solcher besagten Wirkungsquerschnitte in gravothermale Fluid-Ansätze. Hierzu werden verschiedene Literaturansätze mit gleicher physikalischer Basis, aber unterschiedlicher Implementierung analysiert. Dies dient dazu, die eigenen Ergebnisse im Vergleich argumentativ als plausibel zu validieren. Dieser Schritt ist besonders notwendig, da im direkten Vergleich mit N -Körper-Daten signifikante Unterschiede für alle untersuchten Literaturmodelle festgestellt wurden. Die methodische Basis dieser Arbeit führt gründlich in die Modelle und ihre Unterschiede ein. In ihrer Genauigkeit lässt die Analyse einen zentralen Schluss offen: Entweder liegt ein systematischer Fehler in der zugrundeliegenden Datenbasis vor, was als äußerst unwahrscheinlich betrachtet wird, oder die in der Literatur etablierte Basis für die Kopplung von Phasenraum-Dynamik und Fluid-Modellen ist zu stark simplifiziert.

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List of Acronyms

CMB cosmic microwave background. 2, 4
 Λ CDM Λ -Cold Dark Matter. 1, 2, 4, 5
DM Dark Matter. 1–6, 10, 11, 13, 16, II, IV
CDM Cold Dark Matter. 1, 2, 4–6
SIDM Self-Interacting Dark Matter. 1, 2, 5–11, 33, I, II
vdSIDM velocity-dependent Self-Interacting Dark Matter. 1, 8, 13, 15
NFW Navarro–Frenk–White. 4, 5, 9, 14, 20, II, IV
LMFP Long Mean Free Path. 11, 14, 24, 31, 32
SMFP Short Mean Free Path. 12, 14, 24, 31

1 Introduction

In the field of cosmology, which is concerned with understanding the origin, evolution, structure, and ultimate fate of the Universe [1], a significant area of research concerns the cosmological implications of DM. In the search for a theoretical model of DM, a few promising candidates have emerged, with a common assumption being Cold Dark Matter (CDM) used as a basis for Λ -Cold Dark Matter (Λ CDM). Having success on large scales but facing challenges on small scales, a further development of the CDM model, the SIDM was introduced by Spergel and Steinhardt [2], sparking substantial research that highlights SIDM as a promising solution to these challenges.

More recent research goes a step further than simply assuming a constant cross-section for SIDM by extending the self-interaction to velocity dependencies (velocity-dependent Self-Interacting Dark Matter (vdSIDM)). This approach has proven particularly successful in reconciling DM constraints across different physical scales, as it naturally accommodates the high scattering cross-sections inferred in low-velocity environments like dwarf galaxies while suppressing interactions at the higher relative velocities characteristic of galaxy groups and clusters [3, 4].

While SIDM gravothermal fluid descriptions are well known and have been validated for constant cross-sections, as derived by Balberg, Shapiro, and Inagaki [5], gravothermal fluid descriptions in the vdSIDM context have only been published in recent years. The most well-known descriptions are those by Yang and Yu [6], Outmezguine et al. [7], or Yang et al. [8]. These may be by now established, but have never been tested against N -body data using velocity-dependent cross-sections.

Therefore, this thesis focuses on the applicability of gravothermal fluid approaches for velocity-dependent SIDM, providing a systematic comparison between the three approaches. Furthermore, it presents a direct comparison with N -body simulations by Fischer, Yu, and Dolag [9]. The objective is to ascertain whether these fluid descriptions are adequate or require further refinement.

With this goal in mind, work is structured as follows: After starting with a general overview of the theoretical background from Λ CDM to vdSIDM, the methodology is introduced, which provides insight into N -body simulations and gravothermal fluid approaches from mathematical and technical perspectives and concludes with a brief explanation of the main differences and description of the three vdSIDM descriptions. Next, we focus on the mathematical description of the functions underlying the velocity dependence, and then turn to the numerical implementation explaining and verifying the simulation setup finally in the results part, a methodological comparison is conducted between all three models and the N -body data. Lastly, the discussion section provides a comprehensive overview of the findings and offers a range of potential explanations for any observed deviations and the limitations of the study.

2 Theoretical Background

This chapter elaborates the theoretical foundation of this work. It reviews our present-day Standard model of cosmology and motivates the existence of DM. We further introduce the idea of SIDM and discuss the physical evolution of an astrophysical object called SIDM-halo.

2.1 Λ CDM model

The currently accepted standard model of cosmology, known as the Λ CDM, describes an almost flat, expanding Universe, whose energy density is dominated by two components dark energy (Λ) and CDM [10, Chapter 1].

Build upon the cosmological principle, which assumes the Universe to be homogeneous and isotropic on sufficiently large scales [1, Chapter 2.2], its spacetime can be classified by its curvature (zero, positive, negative) [10, Chapter 2]. Measurements of the cosmic microwave background (CMB) indicate that spatial curvature is consistent with zero, thus confirming a nearly flat geometry [11, Chapter 7.3].

While the CMB observations tightly constrain the intrinsic spatial geometry of the Universe (i.e., its spatial curvature), its cosmic evolution is characterized by the expansion of space itself. Observations show that spectral lines of distant galaxies are systematically redshifted, indicating that galaxies recede from one another [1, Chapter 2.3].

Finally, understanding the cosmic evolution requires accounting for non-visible components. In the framework of standard cosmology, the expansion dynamics of the Universe are governed by its intrinsic spatial curvature and its total energy content, which includes radiation and baryonic matter. However, measurements of the CMB establish a spatially flat geometry ($k = 0$), while observations of distant Type Ia supernovae demonstrate that current cosmic expansion is accelerating [10, Chapter 1]. Reconciling these observations with astrophysical gravitational effects on galactic and extra-galactic scales requires a budget dominated by non-visible components, motivating the introduction of non-luminous DM and dark energy [10, Chapter 1].

2.2 Dark Matter

While Section 2.1 established the broader cosmological framework, this thesis now focuses specifically on the DM energy component to align with the central research objectives.

2.2.1 Evidence for Dark Matter

In a historical overview, Bertone and Hooper [12] pointed out that the concept of unseen mass, such as faint stars, cold gas clouds, or compact remnants, was already considered in the 19th century, long before the modern physical understanding of non-baryonic DM emerged. Robust dynamical evidence for missing mass only developed in the twentieth century, when various astrophysical observations began to indicate significantly more mass than previously expected and that could be accounted for by visible baryonic matter alone [12, Section II.C].

Galactic Rotation Curves

Early indications on the scales of galaxy clusters were provided by Fritz Zwicky [13], who compared observational data of the coma cluster to its corresponding theoretical predictions (using the viral theorem), which showed that its dynamical mass far exceeded its visible mass [12, p. 6].

Further work supporting the evidence for DM extends to the galactic scales, exploiting that on this particular scale mass distributions can be directly probed via rotation curves: If the galactic mass were solely made up of big visible matter, the velocity profile $v(r)$ would decline as $r^{-1/2}$ at large radii. Observations by Freeman [14], which were later systematically confirmed by Rubin and Ford [15], reveal however flat rotation profiles.

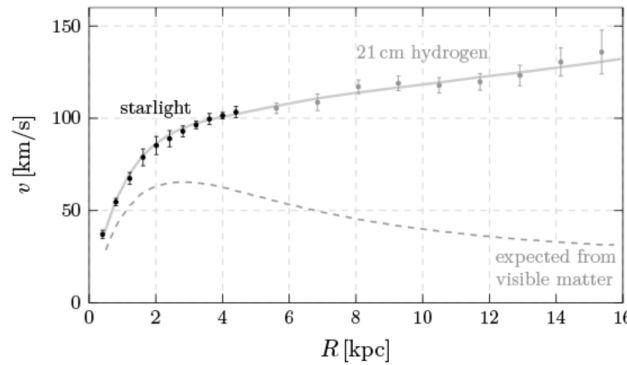


Figure 2.1: Expected Keplerian rotation profile vs. the observed flat rotation curve, indicating an extended DM-halo [10, Fig. 2.9].

As shown in Figure 2.1, this discrepancy indicates the presence of an extended, non-luminous halo surrounding the galaxy, consisting of some yet unknown form of matter.

Gravitational Lensing and the Cosmic Microwave Background

Additional evidence for DM is provided by observations of gravitational lensing events. According to general relativity, massive objects bend the path of light traveling through spacetime. As a result, distant light sources appear distorted or multiple images of the same source are observed. Analyzing these lensing effects allows

for the reconstruction of the total mass distribution in galaxies and galaxy clusters. In many cases, the inferred mass significantly exceeds the amount of visible matter, pointing to a substantial DM component [1, Chapter 7.4].

Independent cosmological evidence for DM is provided by measurements of the CMB. The observed temperature anisotropy spectrum depends sensitively on the matter and energy content of the universe and cannot be explained by baryonic matter alone. Observations by the Planck mission support the presence of CDM and allow its density parameter Ω_{cdm} to be determined with high precision [11, Section 3.5].

Further Evidence

The observations discussed above represent only some of the most important lines of evidence for the existence of DM. Additional support emerges from the formation of the cosmic large scale structure, where observations of galaxy clustering and cluster counts align precisely with the predictions of a DM-dominated Universe [12, 16].

2.2.2 Cold Dark Matter and its Small-Scale Challenges

Given the overwhelming observational evidence across multiple astrophysical scales, the existence of DM is firmly established in the standard cosmological paradigm. The model underlying most descriptions is CDM. It describes non-interactive particles that were non-relativistic during structure formation, enabling density perturbations to grow efficiently into the large scale structures observed today [10, Chapter 2.3.2]. Despite its remarkable success in describing the large scale structure of the universe, CDM faces several discrepancies between simulations and observations on galactic and sub-galactic scales. In the following, the most prominent DM-related challenges are briefly introduced. A comprehensive review of these issues is provided by Bullock and Boylan-Kolchin [17].

Missing Satellites Problem

Satellite galaxies are dwarf galaxies which are gravitationally bound to a larger host and orbiting within its DM-halo. The *missing satellites problem* refers to the severe tension between the hundreds of subhalos predicted by high-resolution CDM simulations for Milky-Way-sized halos and the significantly smaller number of actually observed satellite galaxies [18, 19, 16].

Core-Cusp Problem

Numerical Λ CDM simulations predict that DM-halos follow the NFW density profile [20],

$$\rho(r) = \frac{\rho_s}{\left(\frac{r}{r_s}\right) \left(1 + \frac{r}{r_s}\right)^2}, \quad (2.1)$$

where ρ_s and r_s are the so called characteristic density and the scale radius of the halo, respectively.

The NFW profile exhibits a steep increase in density towards the center of the halo, resulting in a so-called *cusp*. While the latter is generally consistent with observations of large galaxies and galaxy clusters, observations of dwarf and low-surface-brightness galaxies often indicate a nearly constant central density, referred to as a *core* [21].

Too Big to Fail Problem

The *too big to fail problem* extends the missing satellites challenge from pure subhalo counts to the internal kinematics of DM subhalos [22].

Cosmological Λ CDM simulations predict a population of massive and dense subhalos surrounding Milky-Way-like galaxies that should be dynamically “too big to fail” to host star formation. Owing to their deep gravitational potential wells, these subhalos ought to retain gas and form stars efficiently. However, their predicted central mass densities, inferred from circular velocity profiles $v_{\text{circ}}(r)$, are systematically higher than those derived from the stellar kinematics of the brightest observed dwarf spheroidal galaxies [22].

2.3 SIDM

So far, it has become clear that CDM is a highly successful model on large scales. However, on small scales, significant challenges remain. In this light, Spergel and Steinhardt [2] proposed SIDM as an alternative that maintains the successes of CDM on large scales while simultaneously addressing its small-scale issues.

2.3.1 Motivation for SIDM

The framework proposed by Spergel and Steinhardt [2] limits the mean free path of DM particles to a range between approximately 1 kpc and 1 Mpc in high-density environments.

In dense galactic centers, frequent elastic self-scattering would trigger an efficient heat transfer that redistributes kinetic energy, converting the central cusp into a flat core and thus resolving the core-cusp problem [2, 23]. Simultaneously, this central mass reduction flattens the inner rotation curves and lowers the predicted central velocity dispersions, inherently mitigating the too big to fail problem [24, 25]. Furthermore, self-interactions enhance the sensitivity of subhalos to tidal stripping and disruption, providing a natural mechanism to reduce substructure abundance and relieve the missing satellites problem [24, 16].

2.3.2 Basic Principles of DM Self-Interactions

To capture the macroscopic effects of DM self-interactions, galactic halos are described using the statistical mechanics of self-gravitating systems. The state of a halo is fully encapsulated by its continuous phase-space distribution function $f(\vec{x}, \vec{v}, t)$, which provides the foundation for both fluid descriptions and particle-based simulation approaches.

The key information of macroscopic stellar and DM systems can be derived by integrating their phase-space distribution functions [26, Chapter 4]. The fundamental evolution of $f(\vec{x}, \vec{v}, t)$ in position and velocity space is governed by the collisional Boltzmann equation [26, Chapter 7]

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} - \frac{\partial \Phi}{\partial \vec{x}} \cdot \frac{\partial f}{\partial \vec{v}} = \left(\frac{\partial f}{\partial t} \right)_{\text{coll}}, \quad (2.2)$$

where Φ denotes the self-consistent global gravitational potential. While the standard collisionless CDM paradigm assumes that the collision term on the right-hand side vanishes identically ($(\partial f / \partial t)_{\text{coll}} = 0$), the core concept of SIDM is reflected in the introduction of a non-zero collision operator $(\partial f / \partial t)_{\text{coll}} \neq 0$.

This treats the inner halo as an isolated, self-gravitating gas trapped within its own potential well. Macroscopically, the random thermal motion of individual particles generates a kinetic pressure that counteracts the gravitational collapse, allowing thermodynamic principles to drive the structural evolution of the halo [5].

The mathematical description for the simulation case will be elaborated in Chapter 3; the foundational physical parameters of the SIDM framework are defined below.

Scattering Cross-Section

To quantify the self-interaction probability between DM particles, the framework introduces the mass-normalized scattering cross-section σ/m representing the effective physical interaction area per unit mass [27, Chapter 7.3.6]. While $\sigma/m \equiv 0$ is strictly valid for collisionless CDM, a non-zero value establishes a finite probability for elastic self-scattering processes.

Velocity Dispersion and Analogue Temperature

In galactic dynamics, the velocity dispersion σ_v^2 measures the kinetic energy distribution of particles within a system. In a collisional DM-halo, this quantity reads

$$\sigma_v^2 \propto k_B \frac{T}{m} \quad (2.3)$$

and acts as the macroscopic analogue to thermodynamic temperature [16].

The Mean Free Path

While the cross-section determines the micro physical interaction probability, the macroscopic collision frequency depends on the local density. The mean free path λ represents the average distance a particle travels between collisions and is defined as [28, eq. 31]

$$\lambda = \frac{1}{\rho(r)(\sigma/m)} \quad (2.4)$$

where $\rho(r)$ is the local mass density profile and σ/m the aforementioned scattering cross-section.

2.3.3 Evolution of SIDM Halos

The structural evolution of SIDM halos proceeds through two distinct phases [5]. In the initial *core formation phase*, a steep gradient in the velocity dispersion σ_v drives heat inward via frequent self-scatterings. This energy influx expands the central potential well, flattening the initial density cusp to an isothermal core by the maximum expansion time t_{cf} . Once the core becomes isothermal, the heat flow reverses. The core transfers thermal energy to the cooler outer halo and subsequently contracts. Consequently, the central density ρ rises and the mean free path λ decreases, which sharply accelerates the collision rate. This runaway thermal instability triggers the *gravothermal core-collapse phase*, driving the central density to extreme values by the collapse time t_{cc} .

2.3.4 Velocity-Dependent Self-Interactions

Although a constant cross-section σ/m mitigates small-scale issues in dwarf galaxies, observational constraints across broader astronomical scales require a velocity-dependent cross-section, $\sigma = \sigma(v)$ [16]. Because high-velocity collisions in massive galaxy clusters are strongly suppressed compared to low-velocity encounters in dwarf galaxies, an accurate multi-scale SIDM model requires a cross-section that scales inversely with the relative collision velocity.

3 Simulation Methods

Over the years, various methods have been developed to simulate and analyse SIDM. The two most established approaches are N -body simulations and fluid simulations. Newer methods include, for example, the parametric model proposed by Yang et al. [29].

This Chapter focuses on the two approaches. It provides a brief introduction to N -body simulations as well as a short overview of the data that will be used later on. This is followed by a detailed explanation of the gravothermal fluid equations. Lastly, an overview of the three methods of implementing vdSIDM in the gravothermal fluid equations is presented.

3.1 Scaling and Phase Definitions

Let us first start by introducing the following definition. Generally tracking the theoretical core-collapse singularity at which the central density diverges ($\rho_c \rightarrow \infty$) is computationally unfeasible for both simulation types. To establish an unambiguous criterion for comparative analysis, the absolute core-collapse time t_{cc} is defined via a density-driven threshold based on a predefined scaling factor ζ ,

$$\rho_c(t_{cc}) = \zeta \cdot \rho_c(t = 0). \quad (3.1)$$

Complementary to this final collapse boundary, the initial *core formation phase* is defined as the evolutionary period up until t_{cf} , where the expanding central core reaches its absolute minimum density value ρ_{cf} . This marks the turnaround point physical transition before the structural gravothermal runaway phase initiates [28].

To enable a synchronized, cross-model comparison that isolates the runaway collapse trajectory across varying structural scales and cross-sections, the temporal axis is normalized using a shifted and dimensionless time scale τ ,

$$\tau = \frac{t - t_{cf}}{t_{cc}}. \quad (3.2)$$

3.2 N -body Simulations

The fundamental concept of N -body simulations is to sample the continuous phase space distribution function $f(\vec{x}, \vec{v}, t)$ using a discrete ensemble of N point masses rather than treating it as a continuous fluid. Mathematically, this discretization reduces the multi-dimensional partial differential Boltzmann equation to a system of N coupled, ordinary differential equations. For every particle i , the numerical solver computes the gravitational acceleration resulting from all other particles within the

system [26, Chapter 3],

$$\frac{d^2 \vec{x}_i}{dt^2} = -G \sum_{\substack{j=1 \\ j \neq i}}^N m_j \frac{\vec{x}_i - \vec{x}_j}{|\vec{x}_i - \vec{x}_j|^3}. \quad (3.3)$$

In the context of SIDM, these particles are assigned specific scattering cross-sections, enabling them to undergo localized, stochastic elastic collisions alongside their smooth integration.

3.2.1 N -body Data and Numerical Implementations

Some of the most recent and rigorous SIDM simulations have been developed by Fischer, Yu, and Dolag [9], utilizing `OPENGADGET3`. Their computational framework has been rigorously tested and verified over several years. Furthermore, their most recent results have been cross-evaluated by Ulrich [30] using the `KISS-SIDM` code.

To model a dark-matter-only halo, the setup by Fischer, Yu, and Dolag [9] uses initial conditions generated via the module `SphericIC` which is based on a NFW profile and motivated by the perturber of the stellar stream GD-1. The halo is sampled up to a radius of $r_{\text{cut}} = 10 r_s$ or $15 r_s$, with particle escape prevented by an elastic spherical boundary condition at 200 kpc. Initial velocities are derived using an Eddington inversion method (see sec 6.2.2).

To ensure strict convergence during late-stage runaway collapse, the code performs a systematic variation of the gravitational time-step parameter η and a targeted SIDM time-step parameter τ under a minimum threshold Δt_{min} , while enhancing central force resolution via a precise tree opening angle of $\alpha = 5 \times 10^{-4}$. Maintaining a global energy conservation error well below 1% is strictly mandatory to prevent artificial acceleration of the gravothermal collapse.

Crucially, despite starting in perfect velocity isotropy, the system develops a radial velocity anisotropy during the late-stage core collapse, quantified by the parameter

$$\beta_{\text{aniso}} = 1 - \sigma_t^2 / (2\sigma_r^2) \quad (3.4)$$

where σ_t and σ_r are the tangential and radial velocity dispersions. This directional imbalance represents a subtle physical limitation, as the subsequent macroscopic fluid equations rely on the assumption of velocity isotropy.

Lastly, the simulations solely consider the dynamics of dark matter particles, meaning that effects such as baryonic contraction are neglected. Furthermore, tracking the final singularity remains resolution-limited by the finite particle mass. In addition, particles associated with the high-velocity tail of the phase space distribution cause a small fraction of particles to escape to large radii, requiring a strict control via artificial boundary conditions to avoid numerical round-off errors in double-precision gravity calculations.

3.2.2 Post Processing

When analyzing the resulting physical profiles from simulations by Fischer, Yu, and Dolag [9], discrete numerical errors as well as stochastic noise must be carefully managed. Due to finite particle resolution, the innermost regions suffer from artificial relaxation, requiring post-processing smoothing via radial binning.

To ensure a reliable quantitative comparison of the central halo evolution and to mitigate numerical noise in the innermost region of the discrete N -body data, we proceed as follows: The first four radial bins (0 - 3) of the N -body snapshots are excluded due to resolution limits, and the inner core density is defined as the average over the bins 4 to 12, which safely reside within the physical core radius. For the fluid frameworks, a corresponding spatial interpolation of the density profile $\rho(r)$ is performed at each discrete time step, evaluating the continuum solution at a fixed physical radius to match the localized spatial scale of the N -body data.

3.3 Gravothermal Fluid Description

Complementary to N -body simulations, the gravothermal fluid description provides an alternative framework to model the long-term evolution of SIDM-halos. This approach was pioneered by Balberg, Shapiro, and Inagaki [5], building upon the classical thermodynamic principles of self-gravitating stellar systems originally developed by Lynden-Bell and Eggleton [31]. It was further advanced and numerically evaluated for high-mass regimes by Pollack and Steinhardt [28]. In later sections of this work, the open-source code provided by Nishikawa, Boddy, and Kaplinghat [32] is used, which will henceforth be referred to as the `GravothermalSIDM` code.

3.3.1 The Gravothermal Fluid Equations

Within the gravothermal fluid approximation, the macroscopic evolution of an isolated, spherical, virialized SIDM-halo is described by a system of four coupled differential equations. These were first extended to DM-self-interactions by Balberg, Shapiro, and Inagaki [5] and read

$$\frac{\partial M}{\partial r} = 4\pi r^2 \rho, \quad (3.5)$$

$$\frac{\partial(\rho\nu^2)}{\partial r} = -\frac{GM\rho}{r^2}, \quad (3.6)$$

$$\frac{\partial L}{\partial r} = -4\pi r^2 \rho \nu^2 \left(\frac{\partial}{\partial t} \right)_M \ln \left(\frac{\nu^3}{\rho} \right), \quad (3.7)$$

$$\frac{L}{4\pi r^2} = -\kappa \frac{\partial T}{\partial r}, \quad (3.8)$$

where $M(r)$ represents the enclosed mass within the radius r , $\rho(r)$ is the corresponding mass density, $\nu(r)$ denotes the one-dimensional velocity dispersion acting

as the fluid temperature analogue see sec 2.3.2, $L(r)$ is the heat luminosity, T is the temperature, and κ the thermal conductivity of the DM fluid.

Integrating the phase space distribution function over velocity space,

$$\rho = m \int f d^3v \quad (3.9)$$

defines the local mass density. In the case of spherical symmetry, the enclosed mass

$$M(r, t) = 4\pi \int_0^r \rho(r', t) r'^2 dr' \quad (3.10)$$

directly yields Equation (3.5), expressing mass conservation as the zeroth velocity moment of the Boltzmann equation. By combining the first velocity moment under spherical symmetry, an isotropic velocity dispersion and neglecting the bulk motion results in Equation (3.6) (the isotropic Jeans equation) which enforces mechanical equilibrium between inward gravity and outward pressure.

Equation (3.7) encapsulates the first law of thermodynamics. For an ideal monatomic fluid with specific internal energy $u = 3\nu^2/2$, the logarithmic term $\ln(\nu^3/\rho)$ is proportional to the specific entropy. The radial variation of conductive luminosity therefore dictates the temporal entropy evolution of each mass shell. Equation (3.8) provides the constitutive transport closure, relating the total heat flux to the gradient of the effective temperature, $T \propto \nu^2$.

3.3.2 Thermal Conductivity and Transport Mechanism

Unlike collisionless systems, where heat transport is mediated by weak, long-range gravitational encounters [cf. 26, Chapter 7], heat conduction within an SIDM-halo is driven by localized, elastic particle interactions. Consequently, the transport coefficient κ is highly dynamic and depends on the local Knudsen number, defined as the ratio of the mean free path λ_{mfp} to the gravitational scale height H

$$H = \sqrt{\frac{\nu^2}{4\pi G\rho}} \quad (3.11)$$

$$\text{Kn} = \frac{\lambda_{\text{mfp}}}{H} = \frac{1}{\rho(\sigma_{\text{SIDM}}/m)H}. \quad (3.12)$$

The value of Kn fundamentally dictates the geometric regime of the heat conduction, which can be characterized using the local relaxation time, $t_r = 1/(a\rho(\sigma_{\text{SIDM}}/m)\nu)$, where a is a dimensionless constant of order unity [28]. In the diffuse outer regions ($\text{Kn} \gg 1$), the system operates in the Long Mean Free Path (LMFP) regime. Here, heat transport is driven by rare, discrete collisions, meaning the conductivity scales directly with the collision rate and the cross-section,

$$\kappa_{\text{LMFP}} \propto \rho \nu^2 H^2 / t_r \propto \frac{\rho \nu^3}{G} \left(\frac{\sigma_{\text{SIDM}}}{m} \right). \quad (3.13)$$

Conversely, in the ultra-dense collapsed core ($\text{Kn} \ll 1$), the system enters the Short Mean Free Path (SMFP) regime. In this fluid-like limit, frequent scatterings locally thermalize the velocity space and impede long-range energy flow. This results in a Spitzer-like conductivity that is inversely proportional to the cross-section,

$$\kappa_{\text{SMFP}} \propto \rho \nu^2 t_r \propto \frac{\nu}{\sigma_{\text{SIDM}}/m}. \quad (3.14)$$

To smoothly bridge these physical limits within a unified numerical architecture, the effective conductivity κ_{eff} is evaluated using a generalized interpolation scheme, originally introduced by Balberg, Shapiro, and Inagaki [5] and extended by Nishikawa, Boddy, and Kaplinghat [32],

$$\kappa_{\text{eff}} = \left(\kappa_{\text{LMFP}}^{-\alpha} + \kappa_{\text{SMFP}}^{-\alpha} \right)^{-1/\alpha}, \quad (3.15)$$

where α serves as a tuning exponent. While the standard `GravothermalSIDM` code natively enforces the classical linear interpolation ($\alpha = 1$) the quantification of how modifications to the transition zone ($\text{Kn} \approx 1$) impact the core-collapse timescale remains a key research area for cross-model calibration against N -body reference data.

3.3.3 Halo Initialization and Calibration Parameters

To initiate the gravothermal evolution within the `GravothermalSIDM` framework, the initial grid states are configured using a discrete set of variables compiled in Table 3.1.

Category	Parameter	Physical Meaning / Role
Structural Profile	<code>profile</code>	Target spatial density profile configuration
	<code>r_s</code>	Halo scale radius
	<code>rho_s</code>	Central scale mass density at $t = 0$
Particle Physics	<code>sigma_m_with_units</code>	Core elastic scattering cross-section (σ_{SIDM}/m)
	<code>w_units</code>	Characteristic particle velocity scale
Calibration Variables	<code>C</code>	Dimensionless normalization factor
	<code>n_shells</code>	Total number of grid shells
	<code>r_min / r_max</code>	Dimensionless inner and outer physical boundaries of the grid
Numerical Solver	<code>flag_timestep...</code>	Dynamic convergence triggers and adaptive time-stepping controls

Table 3.1: Physical and Numerical Parameters of the Fluid Code Initialization.

These parameters are further classified into four functional groups: First, **Structural Profile Parameters** establish the spatial macro-state geometry at $t = 0$. Second, **Particle Physics Constants** define the fundamental scattering properties that remain active throughout the simulation runtime. Third, the **Calibration Variables** scale the thermal conductivity to directly regulate the timescale of the core collapse, alongside critical grid parameters that must be calibrated to ensure numerical convergence. Although grid sizes and shell numbers are not free physical

parameters, they dictate the resolution boundaries and exhibit strict convergence thresholds that vary depending on the target halo profile. Fourth, **Numerical Solver Settings** control adaptive time-stepping and error triggers; these variables remain fixed to their default configuration unless specified otherwise.

3.4 Velocity-Dependent Cross-Section Integration

As already mentioned in section 2.3.4, vdSIDM is a highly topical area. While this work utilizes the results of vdSIDM N -body simulations from Fischer, Yu, and Dolag [9], separate implementations are carried out for the fluid solution. To this end three common approaches exist in the literature, which share a common theoretical foundation. After first elaborating this fundamental concepts, we proceed by discussing the individual techniques.

Of particular interest is the comparison of results where different implementations have been used for the source model. This not only allows to compare different models with each other but also to analyze how a specific choice of implementation can influence the physical results.

3.4.1 Theoretical Foundation

In the most general kinetic description, independent of any fluid closure approximations, the local interaction rate of a velocity-dependent property $G(v_{\text{rel}})$ requires the evaluation over the full multi-dimensional phase space distribution function $f(\vec{r}, \vec{v}, t)$ [26, cf.],

$$\langle G(v_{\text{rel}}) \rangle = \frac{1}{(\rho(r))^2} \iint f(\vec{r}, \vec{v}_1, t) f(\vec{r}, \vec{v}_2, t) G(v_{\text{rel}}) d^3\vec{v}_1 d^3\vec{v}_2, \quad (3.16)$$

where

$$\rho(r) = \int f(\vec{r}, \vec{v}, t) d^3\vec{v} \quad (3.17)$$

represents the local macroscopic mass density. As the DM-halo evolves and contracts, the true phase space distribution moves away from its initial state.

The standard gravothermal fluid framework inherently requires an isotropic closure condition, assuming that the local particle velocities follow a Maxwell-Boltzmann distribution. Under this assumption, following the methodology of Outmezguine et al. [7], the general phase space integral reduces to a localized thermal average,

$$\langle G(v_{\text{rel}}) \rangle = \int_0^\infty \frac{dv_{\text{rel}}}{\sqrt{4\pi\nu^3}} v_{\text{rel}}^2 \exp \left[- \left(\frac{v_{\text{rel}}}{2\nu} \right)^2 \right] G(v_{\text{rel}}), \quad (3.18)$$

where ν denotes the local one-dimensional velocity dispersion.

Following the semi-analytic framework of Yang and Yu [6] and adopting the notation of Outmezguine et al. [7], a dimensionless factor $K_p(\varphi)$ can be defined to track the

systematic deviation from constant hard-sphere scattering,

$$K_p(\varphi) = \frac{\langle \sigma_{\text{visc}} v_{\text{rel}}^p \rangle}{\sigma_0 \langle v_{\text{rel}}^p \rangle_{\text{hard sphere}}}, \quad (3.19)$$

where

$$\varphi \equiv 4\nu^2/w^2 \quad (3.20)$$

represents the radius-dependent kinematic scale factor.

3.4.2 Effective Cross-Section

The technique introduced by Yang and Yu [6] reduces the continuous velocity dependence of the cross-section $\sigma(\nu)$ into a single effective cross-section σ_{eff} . This approach characterizes the entire halo profile with the help of characteristic structural scales determined exclusively by the initial boundary conditions. Mathematically, the effective cross-section is computed via the equation

$$\sigma_{\text{eff}} = \sigma_0 \cdot K_5 \left(\frac{4\nu_{\text{eff,1D}}^2}{w^2} \right), \quad (3.21)$$

where σ_0 represents the core cross-section normalization and K_5 denotes the fifth-order velocity-averaged fluid moment. The characteristic scale-dependent one-dimensional velocity dispersion $\nu_{\text{eff,1D}}$ is derived directly from the maximum circular velocity V_{max} of the initial isolated system,

$$V_{\text{max}} \approx 1.649 \cdot r_s \sqrt{G\rho_s} \quad (3.22)$$

$$\nu_{\text{eff,1D}} = 0.64 \cdot V_{\text{max}}, \quad (3.23)$$

where r_s and ρ_s correspond to the scale radius and scale density of the initial NFW profile, respectively. While Yang and Yu [6] do not explicitly specify the exact expression used to evaluate V_{max} , this relation naturally follows from the maximum of the NFW circular velocity profile [see, e.g., 33, 34].

While this is a reasonable approach, given that the gravothermal fluid code is highly sensitive to constant cross-sections, it remains unclear whether a single limit will suffice in the case of strong variations in velocity dispersion, and whether the results will fall outside an acceptable margin of error relative to the target.

3.4.3 Dispersion-Dependent Cross-Section

Unlike in Yang and Yu [6], the authors in Outmezguine et al. [7] do not use a constant velocity dispersion reducing it to a single value, but instead treat the velocity dispersion as a function of r for each time step t . It includes an explicitly distinguishes between the LMFP and SMFP scattering regimes to construct a continuous descriptions of the halo's heat conductivity. This approach utilizes, velocity-averaged moments to interpolate the effective heat conduction κ across both dynamic regimes.

Mathematically, the corresponding limiting transport coefficients are formulated as,

$$\kappa_{\text{SMFP}} = \frac{3}{2} b\nu \frac{v^5}{\sigma_0 K_5(\varphi)}, \quad (3.24)$$

$$\kappa_{\text{LMFP}} = \frac{3a_C}{8\pi G} \frac{\sigma_0}{m} \rho \nu^3 K_p(\varphi), \quad (3.25)$$

where b_v and a_C are numerical calibration constants, and $K_5(\varphi)$ and $K_p(\varphi)$ denote the respective dimensionless cross-section moments. Crucially, while in [7] the long-range conduction is strictly fixed to the third velocity moment ($p = 3$), we parameterize the index p to evaluate the generalized structural validity of this interpolation technique for arbitrary moment selections.

It is important to note that the authors explicitly note that this specific multi-regime integration has neither yet been calibrated nor verified against vdSIDM N -body simulation data, leaving a vital quantitative uncertainty in their model assumptions.

For the reasons stated above, from this point on wards, any mention of a radius- and time-dependent cross-section used in the simulation will refer to this section. We will either refer to as a dispersion-dependent cross-section model/approach, or as K_p run/s, where p is replaced by the corresponding model in the formula, e.g., K_5 run/s.

3.4.4 Post-Simulation Profile Time-Rescaling

Yang et al. [8] bridges the gap between the effective and the dispersion-dependent cross-section by simulating with a constant cross-section and then applying rescaling it using the dispersion-dependent terms. By combining this differential time-rescaling approach with the dimensionless parameterization introduced in [7] and defined above, the temporal mapping can be expressed as,

$$dt_{\text{eff}} = \frac{\sigma_0}{\sigma_{\text{const}}} K_p(\varphi_0(t)) dt, \quad (3.26)$$

where σ_{const} is the simulated constant cross-section, and $\varphi_0(t) \equiv 4\nu_0^2(t)/w^2$ tracks the temporal evolution of the central one-dimensional velocity dispersion.

Because this retroactive mapping does not dynamically feed back into the fluid transport coefficients during runtime, it remains inherently less physically precise regarding localized, non-self-similar structural profile changes.

4 Setup Development and Validation

As established in Sec. 2.3.4, robust numerical modeling is essential for velocity-dependent DM scattering cross-sections $\sigma(v)$. While Sec. 3.4 introduced the mathematical framework and mapping procedures for a gravothermal fluid code, this chapter focuses on their analytical execution, calibration, and validation against high-resolution N -body data.

All custom code developed for this implementation is publicly available on GitHub at <https://github.com/LukasLLW/vdSIDM-fluid-vs-nbody.git>.

4.1 Velocity-Dependent Cross-Section Mapping

As established in Sec. 3.4, integrating velocity-dependent cross-sections into fluid frameworks requires calculating their velocity averages. Since this procedure is identical for all models evaluated herein, their general expressions are first derived and following evaluated for each model.

The averaging is performed using the specific viscosity cross-section parametrization from Fischer, Yu, and Dolag [9] to ensure direct comparability with their results

$$\frac{\sigma_{\text{visc}}(v_{\text{rel}})}{m} = \frac{\sigma_0}{m} \left[1 + \left(\frac{v_{\text{rel}}}{w} \right)^2 \right]^{-2}, \quad (4.1)$$

where σ_0/m is the low-velocity limit normalization ($v_{\text{rel}} \ll w$), v_{rel} is the relative interaction velocity of the DM particles, and w denotes the characteristic velocity scale.

4.1.1 Analytical Derivation of Moments

By solving for the corresponding velocity moments of Eq. (4.1), the explicit expressions for the dimensionless cross-section factors K_p are derived, focusing specifically on the orders K_3 , K_5 , and K_7 .

By introducing the coordinate transformation $x \equiv (v_{\text{rel}}/2\nu)^2$, which yields $v_{\text{rel}} = 2\nu\sqrt{x}$ and $dv_{\text{rel}} = \frac{\nu}{\sqrt{x}} dx$, the analytical hard-sphere reference limit ($w \rightarrow \infty$) evaluates directly via the standard Euler Gamma function $\Gamma(n)$,

$$\langle v_{\text{rel}}^p \rangle_{\text{hard sphere}} = \frac{(2\nu)^{p+3}\nu}{\sqrt{4\pi\nu^3}} \int_0^\infty x^{\frac{p+1}{2}} e^{-x} dx = \frac{2^{p+2}\nu^p}{\sqrt{\pi}} \Gamma\left(\frac{p+3}{2}\right). \quad (4.2)$$

Applying the identical substitution to the velocity-dependent viscosity cross-section,

the generalized phase-space moment integral takes the structural form,

$$\frac{1}{\sigma_0} \langle \sigma_{\text{visc}} v_{\text{rel}}^p \rangle = \frac{2^{p+2} \nu^p}{\sqrt{\pi}} \int_0^\infty x^{\frac{p+1}{2}} e^{-x} [1 + \varphi x]^{-2} dx, \quad (4.3)$$

where $\varphi \equiv 4\nu^2/w^2$. By systematically varying the moment index $p \in \{3, 5, 7\}$, these expressions can be solved analytically in terms of the classical exponential integral function $E_1(z) = \int_z^\infty t^{-1} e^{-t} dt$.

For the third velocity moment ($p = 3$), the integration results in

$$\begin{aligned} \frac{1}{\sigma_0} \langle \sigma_{\text{visc}} v_{\text{rel}}^3 \rangle &= \frac{32\nu^3}{\sqrt{\pi}} \int_0^\infty x^2 e^{-x} [1 + \varphi x]^{-2} dx \\ &= \frac{32\nu^3}{\sqrt{\pi} \varphi^2} \left[1 + \frac{1}{\varphi} - \left(\frac{2}{\varphi} + \frac{1}{\varphi^2} \right) e^{1/\varphi} E_1 \left(\frac{1}{\varphi} \right) \right]. \end{aligned} \quad (4.4)$$

For the fifth velocity moment ($p = 5$), an algebraic partial fraction decomposition of the integrand yields:

$$\begin{aligned} \frac{1}{\sigma_0} \langle \sigma_{\text{visc}} v_{\text{rel}}^5 \rangle &= \frac{128\nu^5}{\sqrt{\pi}} \int_0^\infty x^3 e^{-x} [1 + \varphi x]^{-2} dx \\ &= \frac{128\nu^5}{\sqrt{\pi} \varphi^3} \left[\varphi - 2 - \frac{1}{\varphi} + \left(\frac{3}{\varphi} + \frac{1}{\varphi^2} \right) e^{1/\varphi} E_1 \left(\frac{1}{\varphi} \right) \right]. \end{aligned} \quad (4.5)$$

An analogous algebraic reduction leads to the higher-order transport expression for the seventh velocity moment ($p = 7$),

$$\begin{aligned} \frac{1}{\sigma_0} \langle \sigma_{\text{visc}} v_{\text{rel}}^7 \rangle &= \frac{512\nu^7}{\sqrt{\pi}} \int_0^\infty x^4 e^{-x} [1 + \varphi x]^{-2} dx \\ &= \frac{512\nu^7}{\sqrt{\pi} \varphi^4} \left[\varphi^2 - 4\varphi + 6 + \frac{2}{\varphi} - \left(\frac{4}{\varphi} + \frac{6}{\varphi^2} + \frac{2}{\varphi^3} \right) e^{1/\varphi} E_1 \left(\frac{1}{\varphi} \right) \right]. \end{aligned} \quad (4.6)$$

Finally, substituting these independent moment solutions into the baseline definition of $K_p(\varphi)$ (see eq 3.19) and evaluating the respective normalizing Gamma functions yields the closed-form, dimensionless scaling functions that directly modulate the fluid transport coefficients

$$K_3(\varphi) = \frac{1}{2\varphi^2} \left[1 + \frac{1}{\varphi} - \left(\frac{2}{\varphi} + \frac{1}{\varphi^2} \right) e^{1/\varphi} E_1 \left(\frac{1}{\varphi} \right) \right], \quad (4.7)$$

$$K_5(\varphi) = \frac{1}{6\varphi^3} \left[\varphi - 2 - \frac{1}{\varphi} + \left(\frac{3}{\varphi} + \frac{1}{\varphi^2} \right) e^{1/\varphi} E_1 \left(\frac{1}{\varphi} \right) \right], \quad (4.8)$$

$$K_7(\varphi) = \frac{1}{24\varphi^4} \left[\varphi^2 - 4\varphi + 6 + \frac{2}{\varphi} - \left(\frac{4}{\varphi} + \frac{6}{\varphi^2} + \frac{2}{\varphi^3} \right) e^{1/\varphi} E_1 \left(\frac{1}{\varphi} \right) \right]. \quad (4.9)$$

The analytical solutions are implemented in `code/analyse_code/Moments.py`, employing a smooth third-order Taylor-series threshold to prevent low-velocity ($\varphi \rightarrow 0$)

underflows. To validate them, `code/analyse_code/thermal_viscosity_moments.py` evaluates Eq. (4.3) via adaptive SciPy quadrature, outputting high-resolution $K_p(\varphi)$ CSV grids that ensure modular adaptability to arbitrary cross-sections.

4.1.2 Moments and Implementation

To validate the functional behavior of the averaged viscosity moments K_p , the discrete analytical expressions ($p \in \{3, 5, 7\}$) derived in Sec. 4.1.1 are visualized in Fig. 4.1 against a continuous spectrum of numerical integrations spanning the interval $p \in [3, 7]$, alongside the underlying normalized viscosity cross-section profile.

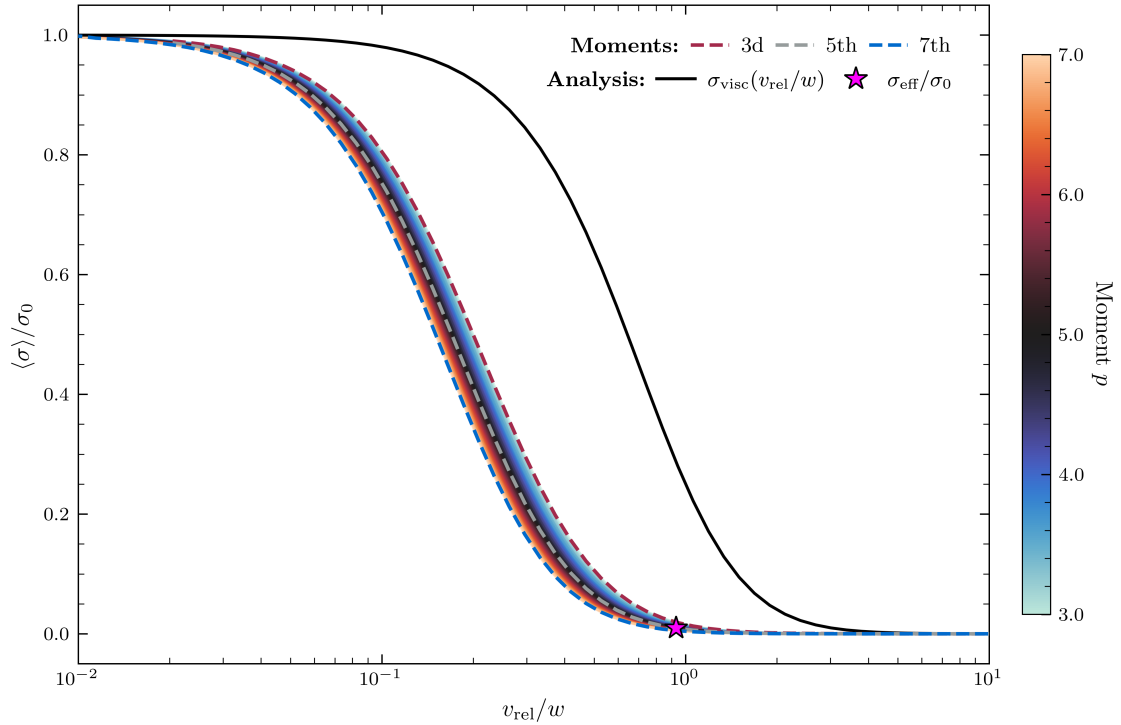


Figure 4.1: Comparison between discrete analytical moments of $p \in \{3, 5, 7\}$ (dashed) and the continuous numerical moment spectrum for $p \in [3, 7]$ (colorful), evaluated at $w = 20$ km/s. The normalized cross-section profile (solid black line, scaled from 0 to 1) is plotted against the velocity ratio v_{rel}/w , while the pink star highlights the local effective cross-section calibrated using r_s and ρ_s .

The scaling curves utilize dimensionless coordinates to illustrate the effective cross-section suppression as a function of the parameter φ (see eq 3.20). The profile of the underlying cross-section is normalized to unity and plotted against the velocity ratio v_{rel}/w . To make contact with physical context, the structural halo parameters r_s and ρ_s are used to determine the local velocity dispersion, enabling the evaluation of the effective cross-section at the characteristic scale of the respective fluid element.

Effective Cross-Section

Following the halo definitions in Sec. 3.4.2, the effective cross-section for the standard fluid approximation is determined via the fifth velocity moment (see eq 3.21). Using the concrete scale parameters $\rho_s = 0.04417 \times 10^{-3} \text{ M}_\odot/\text{pc}^3$ and $r_s = 1.28 \times 10^{-1} \text{ kpc}$ to later on compare with the same setup in [9], the $K_5(\varphi)$ configuration evaluates to:

$$\left(\frac{\sigma_{\text{eff}}}{m}\right)_5 = 0.00945 \times \frac{\sigma_0}{m}. \quad (4.10)$$

For the cross-section normalization of $\sigma_0/m = 6593.89 \text{ cm}^2/\text{g}$ in [9], Eq. (4.10) yields an analytical baseline value of $(\sigma_{\text{eff}}/m)_5 = 62.2 \text{ cm}^2/\text{g}$. Notably, the analytical derivation reveals a systematic arithmetic discrepancy in the phase space averaging presented in [9], which overestimates the effective cross-section at approximately $80 \text{ cm}^2/\text{g}$. This corrected value can subsequently be deployed as a static input parameter for the `GravothermalSIDM` code.

Dispersion-Dependent cross-section

The original `GravothermalSIDM` code exhibits the possibility to extend the dimensionless factors K_p to velocity-depended cross-sections. The code was extended by dynamic callback functions mapping the exact analytical solutions from Eqs. (4.7) - (4.9), alongside a vectorized lookup routine to parse numerical $K_p(\varphi)$ grids directly from external CSV files. The modified version of the scattering initialization function is provided in `GravothermalSIDM/changes.py`.

Post-Simulation Profile Time-Rescaling

To evaluate the post-simulation profile time-rescaling methodology proposed in [8], a post-processing script was developed. This routine ingests the raw evolutionary tracks generated by the effective cross-section configuration of the `GravothermalSIDM` code and performs a retroactive temporal mapping. The scaling transformation is executed based on Eq. (4.3), interpolating structural modifications via the central core properties. The complete Python script handling this automated temporal translation is located in `code/analyse_code/Profile_Time_Rescaling.py`.

4.2 Simulation Setup and Model Calibration

This section outlines the setup for the fluid simulations and N -body data comparison. Using fixed scales (r_s , ρ_s) and varying parameters (σ_0 , w), the setups are checked against the NFW baseline, followed by convergence tests to define stable regimes.

4.2.1 NFW Initial Conditions

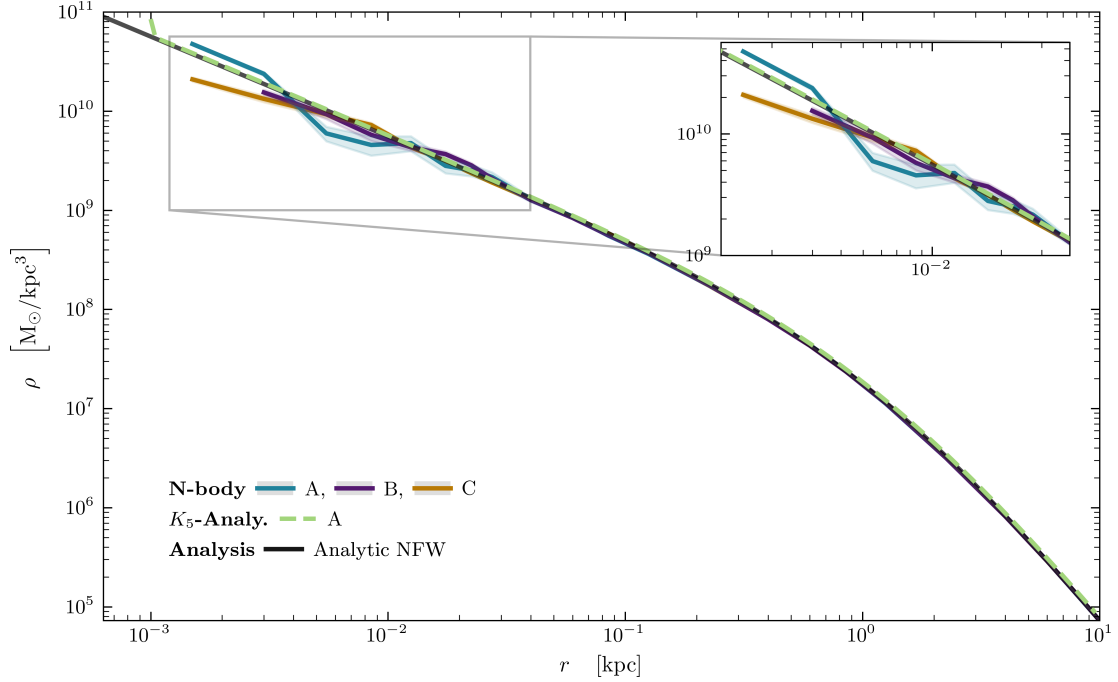


Figure 4.2: Radial density profile $\rho(r)$ at the initial time step ($t = 0$) for the N -body data (runs A, B, and C) and the dispersion-dependent cross-section fluid approach (A), plotted in comparison with the underlying analytical NFW profile.

Fig. 4.2 displays the initial NFW density distribution (cf. Eq. 2.1) for the different simulation setups. The initial structural profile defined by fixed scale parameters is strictly identical across all runs, while the particle physics variables vary between the configurations (see Tab. 4.1). Because all implemented fluid frameworks yield identical initial configurations, the fluid profile in Fig. 4.2 is restricted to the dispersion-dependent cross-section model to preserve visual clarity. This setup includes Run C, which is a direct reproduction of the results from [30] conducted solely to verify that the newly developed code loads and processes the N -body data correctly. Consequently, Run C is omitted from the subsequent evolutionary analysis.

Comparing these configurations against the analytical profile reveals minor initial deviations. Towards larger radii, all profiles show excellent agreement. In the ultra-central regions, however, the discrete N -body data systematically underestimates

Structural		Particle Physics			
Param.	Value	Run	v -dep.	w [km/s]	σ_0/m [cm ² /g]
r_s	1.28 kpc	Sim-A	✓	20	2472.71
ρ_s	$4.417 \times 10^7 M_\odot/\text{kpc}^3$	Sim-B	✓	20	6593.89
		Sim-C	—	—	80

Table 4.1: Overview of the initial structural parameters and run-specific particle physics variables, chosen to match the simulation setup in Fischer, Yu, and Dolag [9].

(Runs B/C) or overestimates (Run A) the density due to finite mass resolution and softening lengths. Conversely, the fluid code exhibits a localized boundary artifact at $r \rightarrow 0$, which remains strictly confined and does not impact the halo evolution. Overall, the initial conditions confirm a correct and validated simulation setup.

4.2.2 Convergence Calibration

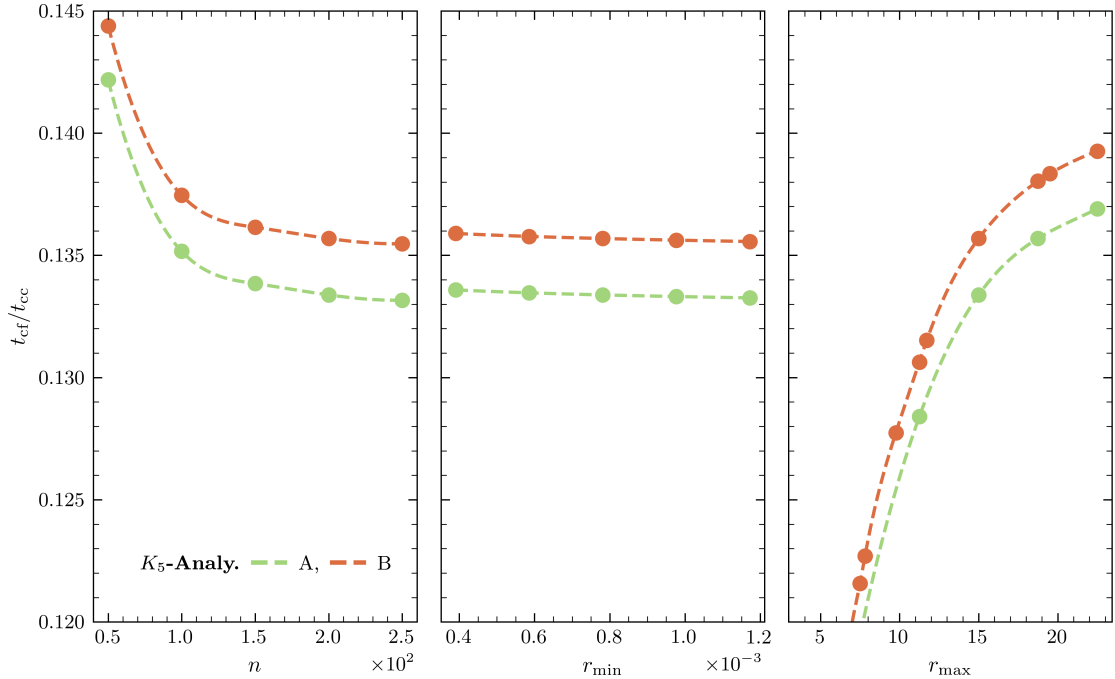


Figure 4.3: Convergence calibration showing the core-collapse time ratio τ as a function of number of shells n , inner and outer boundary r_{min} and r_{max} for the runs A and B.

To guarantee strict numerical stability, a systematic calibration test was performed. The key numerical control variables are varied to monitor the convergence behavior of the characteristic time ratio $\tau_0 = \tau(t=0) = \frac{t_{\text{cf}}}{t_{\text{cc}}}$. The resulting scaling tracks are illustrated in Fig. 4.3. Importantly, the absolute value of τ_0 differs between the

runs *A* and *B*. This variation is intended and aligns completely with theoretical expectations.

Configuration Variable	Symbol	Calibrated Value
Radial Grid Resolution	n_{shells}	200
Outer Boundary Cutoff	r_{max}	16.5
Inner Boundary Cutoff	r_{min}	0.8×10^{-8}

Table 4.2: Optimized parameters determined from convergence calibration in dimensionless Units.

This verification process was executed using the dispersion-dependent cross-section model as the representative baseline. Since the alternative fluid models utilize structurally identical spatial grid structures, their numerical sensitivity to spatial and temporal discretization scales identically. Extending the full multi-parameter grid search across all frameworks was therefore omitted to optimize computational efficiency without loss of physical validity. Based on these stability profiles, the calibrated parameters selected for all production runs are compiled in Tab. 4.2.

5 Results

This chapter validates the integrated gravothermal fluid frameworks against high-resolution N -body data by analyzing initial kinematic states, directly comparing structural profiles, and calibrating the underlying free parameters to map the physical limitations of the fluid approximations.

5.1 Initial Kinematic States and Transport Behaviors

To physically motivate the specific simulation runs, the initial configurations are evaluated. Here, Fig. 5.1 (upper panel) illustrates the radial profile of the initial velocity dispersion $\nu(r)$, derived via the hydrostatic Jeans equation. These kinematic profiles determine the initial velocity-averaged cross-sections $\langle\sigma\rangle$ shown in the lower panel, contrasting the K_5 moment approach and the effective cross-section scaling σ_{eff} .

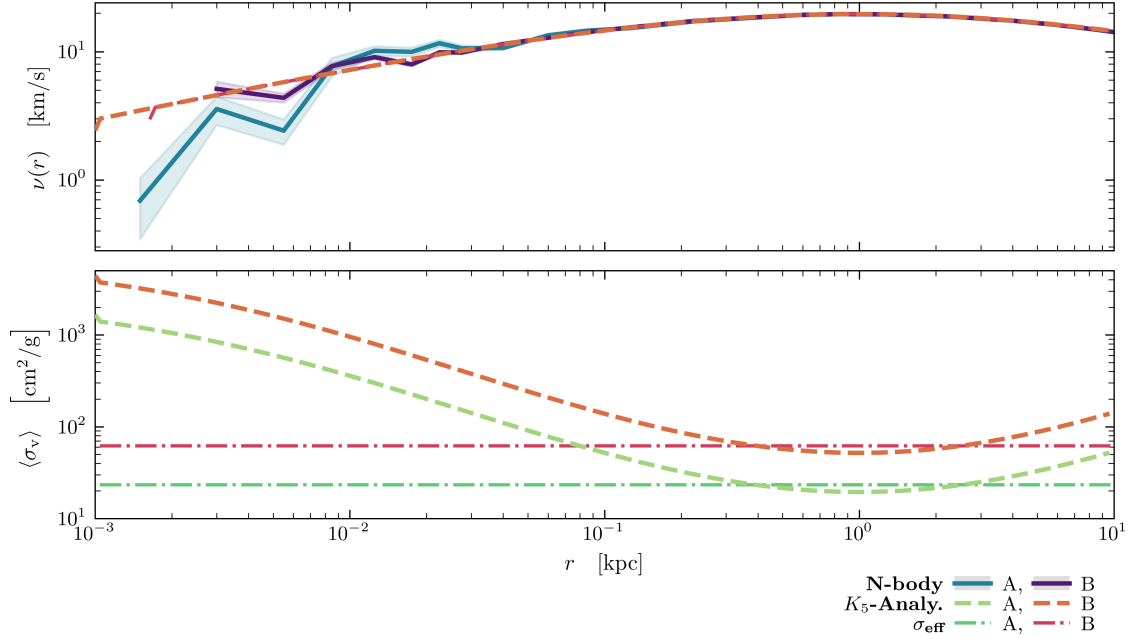


Figure 5.1: Initial radial velocity dispersion $\nu(r)$ (upper panel) validated against N -body data, and the corresponding velocity-averaged cross-sections $\langle\sigma\rangle$ (lower panel) contrasting the K_5 and σ_{eff} models.

While the upper panel presents an actual physical quantity, namely the velocity dispersion extracted directly from the initial N body data for a one to one validation, the lower panel displays model dependent quantities. Because different models treat

the interaction differently, for instance assuming a constant cross-section versus an averaged effective profile, the resulting $\langle\sigma\rangle$ varies by construction. Since it is a priori unknown which effective description best captures the true underlying cross section present in the N -body simulation, no direct N -body comparison is plotted in the lower panel. Instead, this panel serves to illustrate how the choice of model shapes the calculated Kn , directly influencing the subsequent evolutionary dynamics.

These initial states are reflected in the dimensionless Knudsen number Kn (cf. Eq. 3.12), separating the LMFP from the SMFP regimes.

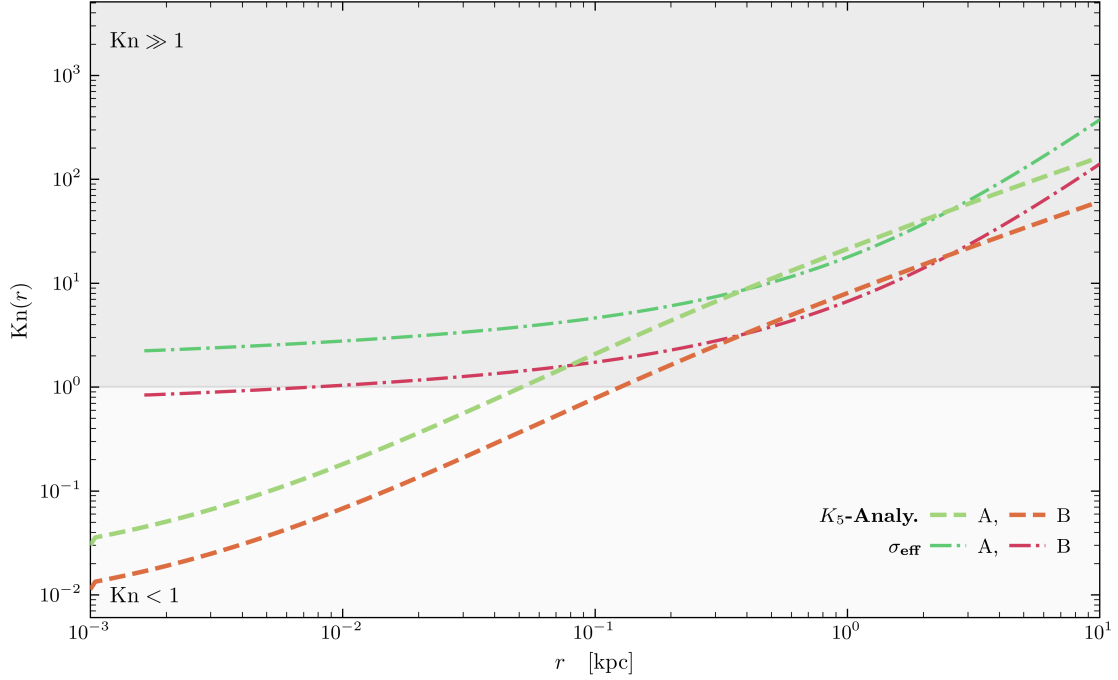


Figure 5.2: Radial distribution of the initial Knudsen number Kn for Simulation A and Simulation B, highlighting transition between the two transport regimes.

The initial Kn -profiles in Fig. 5.2 exhibit structurally identical shapes, but Simulation A is systematically shifted to higher values relative to Simulation B. For the σ_{eff} scaling, Simulation A remains strictly within the LMFP regime ($\text{Kn} > 1$) across all radii, whereas Simulation B crosses into the SMFP domain ($\text{Kn} < 1$) at the innermost radii. In the K_5 -framework, both runs eventually enter the SMFP regime in the vicinity of the halo's core, though Simulation A maintains LMFP conditions over a significantly larger radial extension.

Consequently, Simulation A remains dominated by LMFP relaxation for a longer duration, whereas Simulation B probes the rapid transition into the SMFP regime much earlier in the halo's evolution. So, if a significant difference between A and B were to be observed in dimensionless coordinates τ and ρ/s , one could conclude whether the implementation in SMFP or LMFP is physically incorrect; conversely, if the deviation from the N -body benchmark were the same for both A and B, the underlying concept would likely be flawed.

5.2 Profile Comparisons

To quantitatively assess the temporal and spatial evolution across the simulations, the central core density evolutions are contrasted against the N -body baselines. Fig. 5.3 illustrates the tracking of this inner core density for both Simulation A and B, comparing the high-resolution N -body data against the three ¹ integrated gravothermal fluid models.

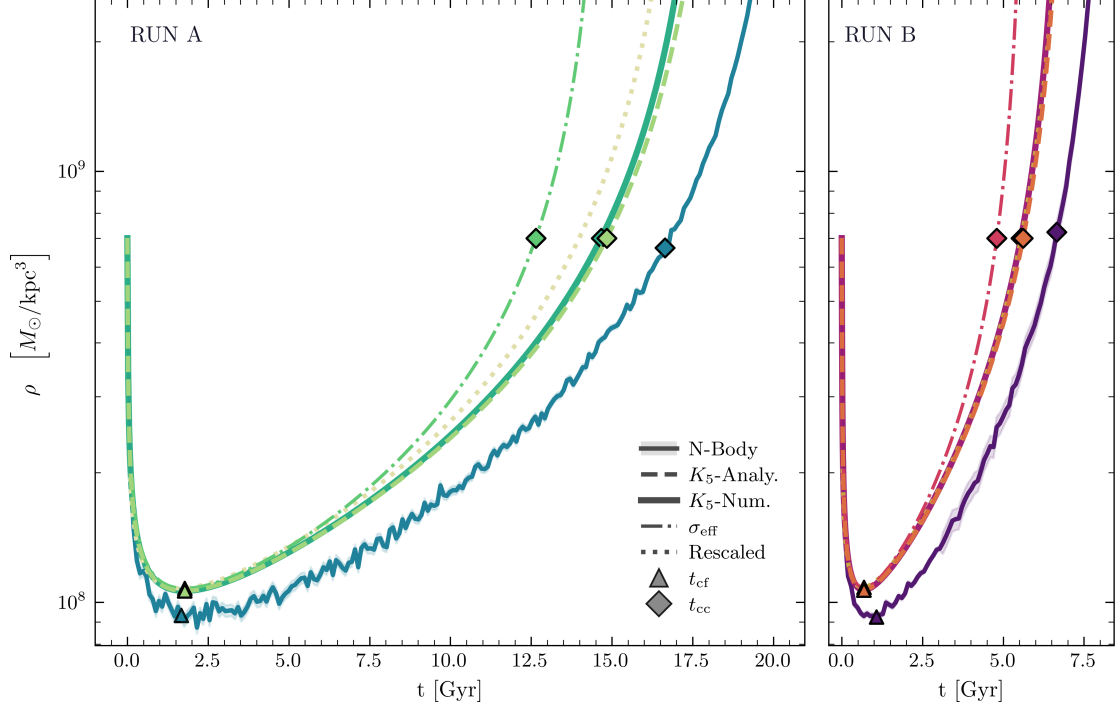


Figure 5.3: Evolution of the inner core density for Simulation A and B, compared to the high-resolution N -body data against three (+1 extra numerical implementation) distinct gravothermal fluid models.

For a direct comparison with the discrete N -body core density (defined via the binning procedure in Sec. 3.2.2), the fluid continuum profiles are spatially interpolated at each time step to extract the local density at the corresponding physical radius. Furthermore, both the time of minimum core radius t_{cf} and the core-collapse time t_{cc} (scaled via $\zeta = 1.05$) are explicitly marked. The numerical extraction and smoothing procedures utilized to isolate these characteristic timescales for both the discrete N -body tracks and the fluid models are detailed in Sec. 3.1.

Several key findings emerge from the evolution shown in Fig. 5.3. Notably, Simulation A exhibits a significantly longer core-collapse timescale ($t_{cc} \approx 15$) compared to Simulation B ($t_{cc} \approx 8$), representing a systematic delay caused by the initial configurations. Furthermore, the ratio t_{cf}/t_{cc} is consistently smaller for Simulation A than for Simulation B. This specific trend holds true across all tested fluid frameworks, despite their varying absolute values. While subtle variations exist among the in-

¹One of the three methods has two implementations (numerical/analytical)

dividual approaches, the effective model (σ_{eff}) deviates most prominently from the standard baselines, a behavior analyzed in deeper detail in Sec. 5.3.

When contrasting the fluid models directly against the high-resolution N -body baseline, distinct physical discrepancies become apparent. Crucially, the core-collapse time t_{cc} occurs significantly later in the N -body data than predicted by any of the fluid models. In contrast, the core-formation timescale t_{cf} exhibits a minor temporal shift between the methods without a clear systematic trend; this characteristic time scale is inherently challenging to extract precisely from the discrete N -body tracks due to stochastic noise during the peak core expansion phase. Furthermore, the minimum core density reached at this turnaround point is substantially lower in the N -body baseline than the values yielded by the idealized fluid equations.

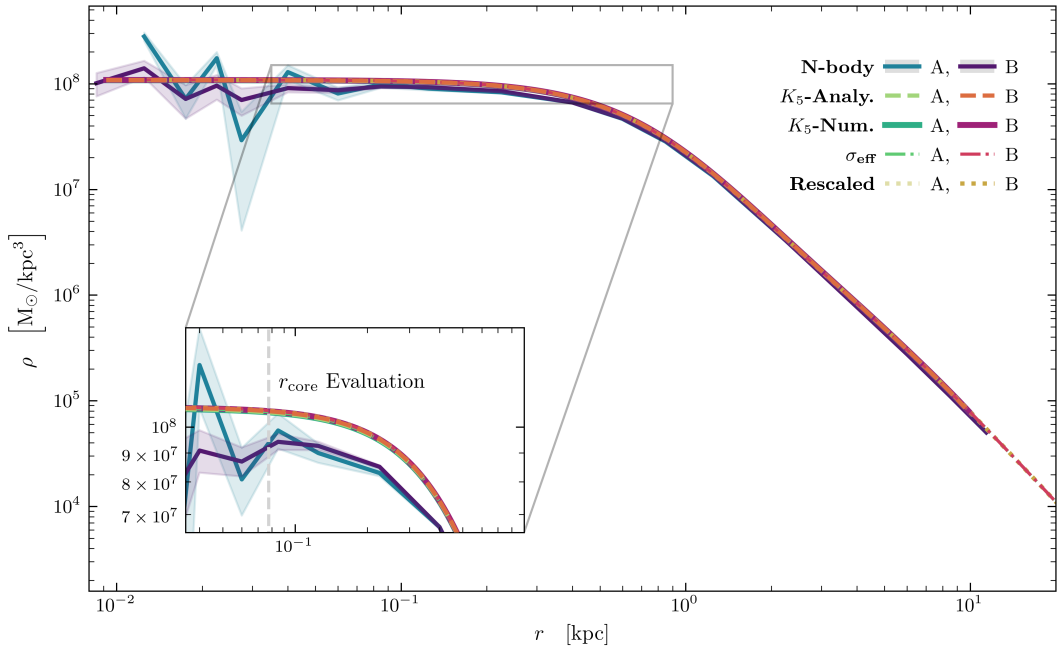


Figure 5.4: Radial density profile comparison highlighting the structural alignment between the fluid models and the N -body baseline snapshot.

To further investigate these temporal offsets, the full spatial snapshots are examined via their radial profiles, evaluated not at identical absolute times but at the equivalent evolutionary milestone t_{cf} . As demonstrated in Fig. 5.4, a distinct structural discrepancy is visible within the inner halo regions. Specifically, when plotted on a logarithmic scale, the central density profiles reveal a clear vertical offset, whereas the outer regions align perfectly across all models.

To conclude the profile analysis, the integrated mass conservation is evaluated in Fig. 5.5. For this comparison, the temporal axis is normalized as t/t_{cc} to align the different evolution timescales. The resulting tracks reveal a significant discrepancy: while the fluid models exhibit perfect mass conservation, the high-resolution N -body data shows a cumulative mass loss of approximately 12% to 15% within the evaluated volume over the total simulation period. Crucially, however, this mass

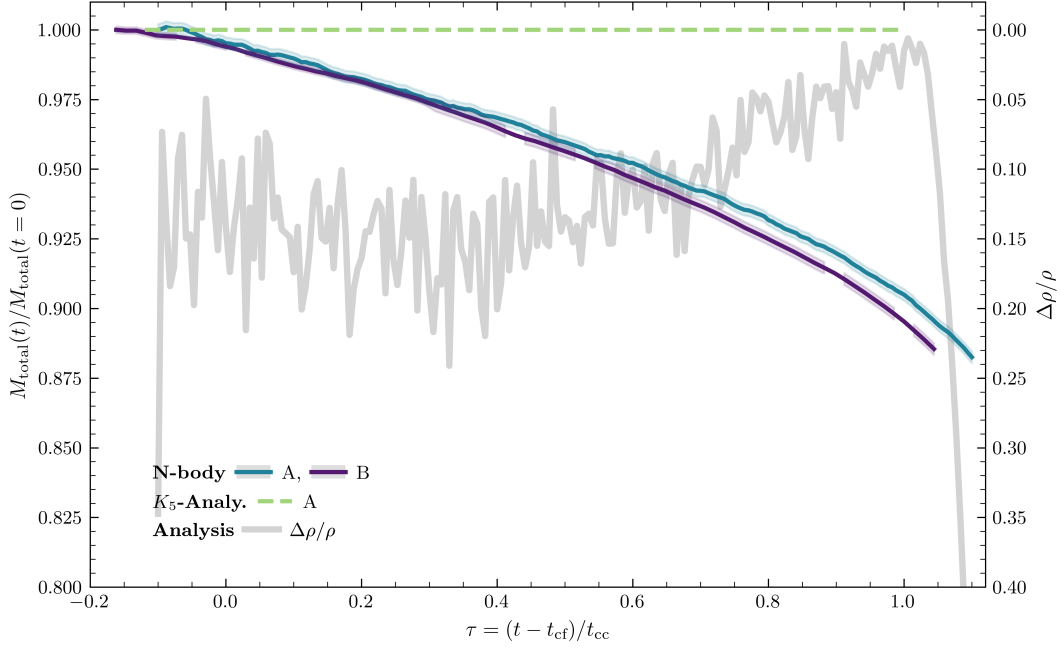


Figure 5.5: Comparison of mass conservation $M_{\text{total}}(t)/M_{\text{total}}(0)$ across different simulation runs alongside the relative density difference $\Delta\rho/\rho = |\rho_{\text{fluid}}(\tau) - \rho_{N\text{-body}}(\tau)|/\rho_{N\text{-body}}(\tau)$ as a function of scaled time τ .

deficit does not resolve the observed offsets, or at best represents only a minor fraction of the underlying issue. This is demonstrated by the gray-shaded curves in the same plot, which track the direct difference between the fluid frameworks and the N -body baselines for the respective configurations between the K_5 fluid model and the N -body track for Simulation A. At the explicitly marked core-formation milestone t_{cf} , the N -body runs have suffered almost no mass loss yet, even though a significant profile deviation is already clearly manifest.

Consequently, while such a mass decline is expected in discrete N -body volumes due to boundary-crossing particles, it cannot account for the early-stage discrepancies. Moreover, this mass loss might specifically reflect a data storage artifact, as radii up to the 200 kpc boundary condition may not be fully included in this specific dataset (see Sec. 3.2.1), unlike in the other runs. This establishes that the observed deviations are not driven by numerical mass non-conservation, motivating a deeper analysis of the underlying fluid implementations.

5.3 Comparison of Fluid Implementations and Code Verification

Since Sec. 5.2 demonstrated a persistent discrepancy between the high-resolution N -body data and all fluid implementations, it is necessary to isolate potential implementation errors from the underlying physics.

Although the high-resolution N -body framework exhibits a localized mass loss (cf. Sec. 5.2), the underlying dataset from Fischer, Yu, and Dolag [9] provides a community-accepted, physically stable benchmark. A systematic investigation or recalibration of the N -body data itself is however outside of the scope of this work.

Consequently, any potential error is assumed within the fluid frameworks. All integrated fluid models rely on the identical mathematical formulation derived for the moments K_p . However, because the analytical and numerical derivations of K_p yield identical results and successfully reproduce the exact reference profiles from Fischer, Yu, and Dolag [9], a fundamental error in this core baseline is highly implausible.

Deviations from K_5 -Analy. (B):

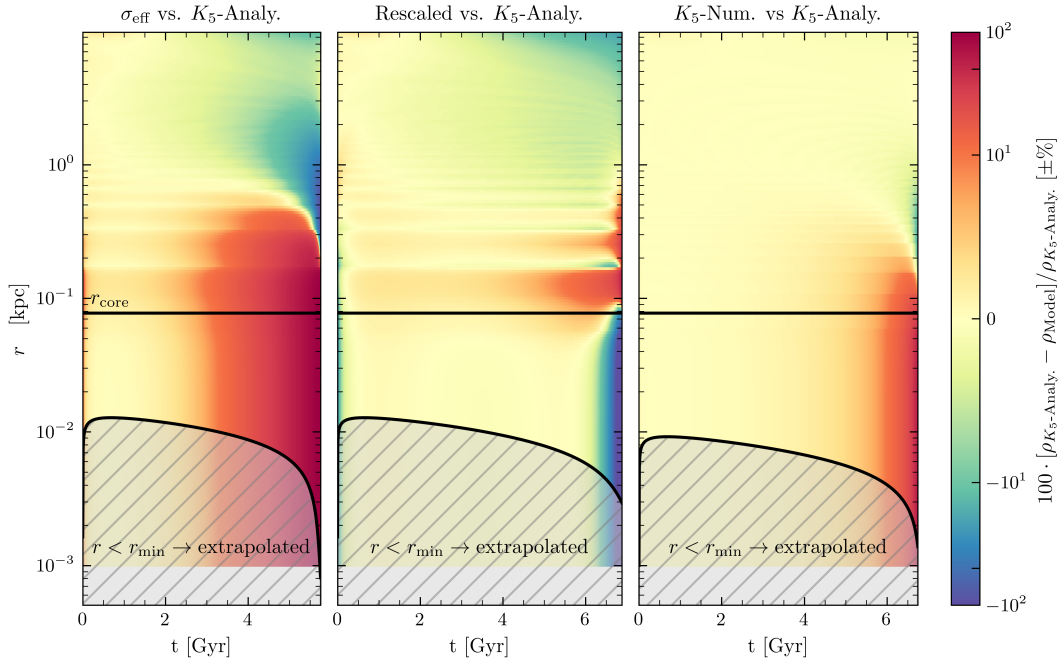


Figure 5.6: Heatmap comparison of the K_5 model against three distinct implementation frameworks for Simulation B.

While the characteristic deviations among the fluid models were already visible within the temporal trajectories of Sec. 5.2, Fig. 5.6 provides a highly detailed heatmap comparison to precisely isolate and visualize where and when the individual implementations diverge from the analytical K_5 baseline. These structurally localized differences are systematically analyzed and discussed below.

5.3.1 Effective vs. K_5 Model

As demonstrated in Fig. 5.6, the effective and Dual-Regime models exhibit excellent structural agreement throughout the majority of the halo evolution. However, the Effective model undergoes an accelerated core-collapse, resulting in a systematically shorter evolutionary timescale and substantial profile deviations emerging around 4 to 5 Gyr.

The underlying physical mechanism driving this divergence is elucidated by the evolution of the velocity-averaged cross-sections illustrated in Fig. 5.7. Although the initial profiles of the averaged cross-sections differ between the frameworks at $t = 0$, they normalize rapidly during the early evolution. Up to the core-formation milestone t_{cf} , the cross-sections of the effective and K_5 approaches nearly coincide. Given that this initial expansion phase occurs rapidly, the effective framework successfully replicates the high-order K_5 behavior at early times.

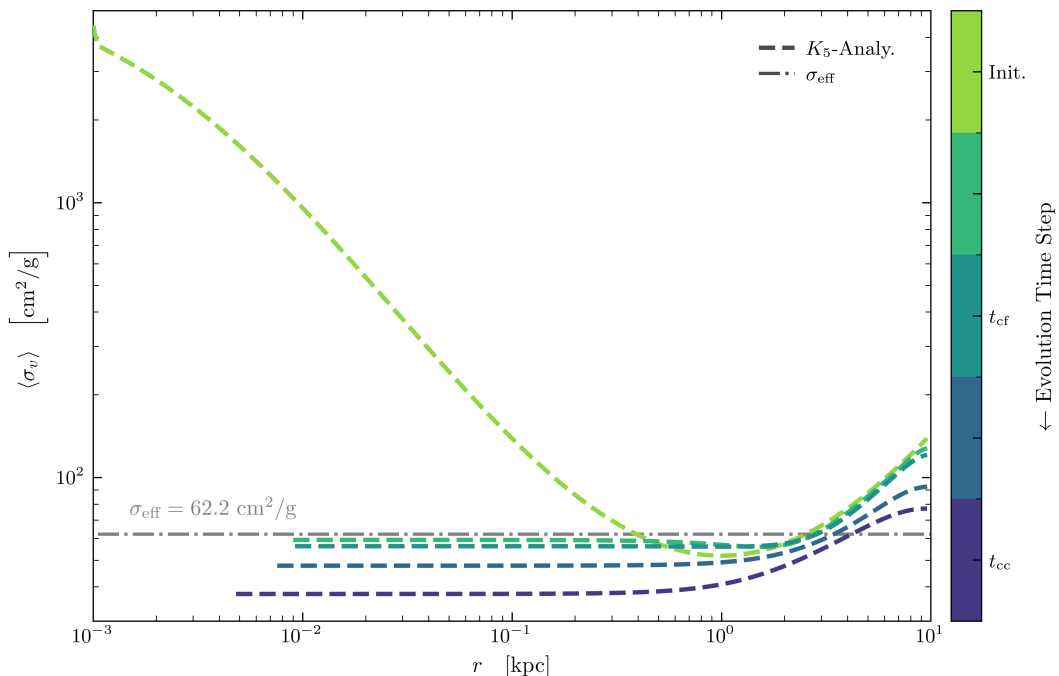


Figure 5.7: Temporal evolution of the radial velocity-averaged cross-sections, comparing the Effective and K_5 frameworks to highlight the divergence driven by late-stage velocity dispersion growth.

In the subsequent collapse phase, however, the continuous growth of the central velocity dispersion induces a pronounced differentiation between the Effective and K_5 cross-sections within the inner halo regions. Because the velocity-averaged cross-section becomes smaller, it results in a systematically slower core-collapse. Consequently, while the Effective model provides a computationally efficient and physically valid approximation during the early evolutionary stages, it fails to capture the late-stage dynamics. Crucially, the strong agreement up to t_{cf} further supports the algorithmic correctness of the implementations, as the subsequent late-time deviations are entirely explained by these distinct physical behaviors.

5.3.2 Rescaling vs. K_5 Model

As illustrated in Fig. 5.6, both models exhibit remarkable agreement throughout the main evolutionary trajectory, with minor discrepancies emerging only after the core-collapse time t_{cc} . This tight alignment provides strong evidence for the algorithmic consistency of the rescaling implementation.

The late-stage deviations stem from the inherent mathematical constraints of the scaling ansatz. Because the framework transforms a static baseline reference curve, it cannot replicate evolutionary behaviors not present in from the original profile. Consequently, during the final collapse phases, the static features of the reference profile dominate the scaling transformation too heavily to maintain precision.

5.3.3 Numerical vs. Analytical K_5 Model

The near-perfect agreement observed between the numerical and analytical K_5 formulations provides definitive validation for the framework. This identical cross-model reproduction eliminates any remaining doubt regarding the numerical integrity of the solver, confirming that the underlying transport equations are solved with high precision and are entirely free of implementation artifacts. Very late deviations can be explained by minute differences (e.g., Taylor approximation; see 4.1.1)

5.4 Calibration Approaches

For the sake of completeness, the following section reviews various calibration methods for gravothermal fluid codes that are well-established in the literature. However, as a definitive resolution to this discrepancy remains highly unlikely within the scope of this work, initially a qualitative approach is adopted to assess how effectively these methods mitigate the differences.

5.4.1 C/α – Analysis

To adjust the structural and temporal evolution in the gravothermal fluid framework, two calibration parameters were tuned: the dimensionless scaling factor C , which predominantly scales the overall evolutionary timescale, and the interpolation parameter α , introduced to bridge the analytically unconstrained transition between the LMFP and SMFP regimes (cf. sec 3.3.2).

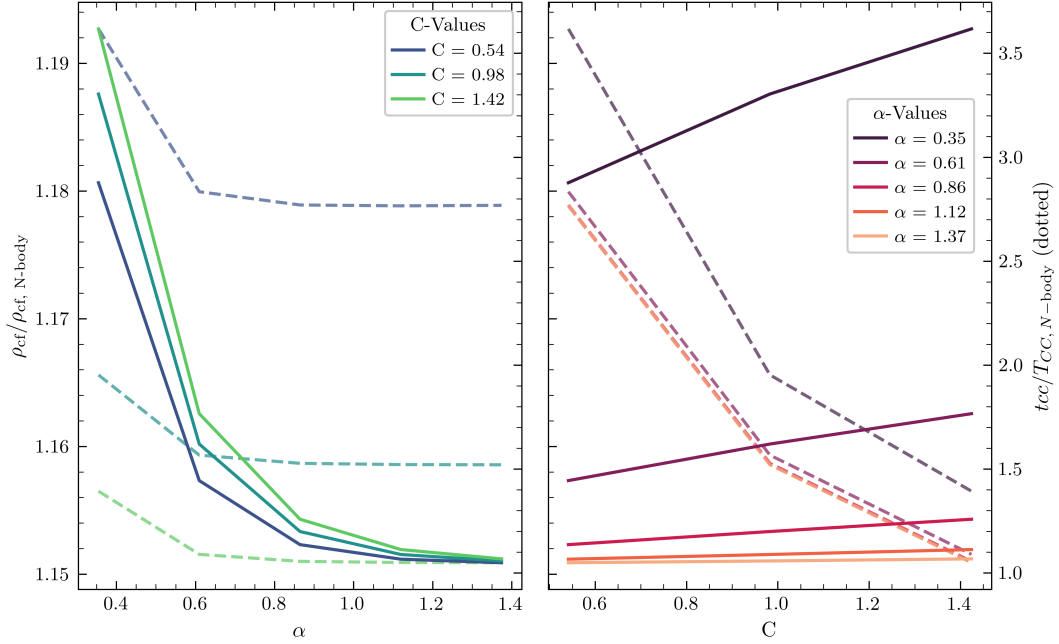


Figure 5.8: Quantitative response of the core-formation timescale t_{cf} and core-collapse timescale t_{cc} relative to the target N -body benchmarks across varying α and C parameter combinations.

Fig. 5.8 makes it immediately clear, within the parameter space, whether the error can be corrected. Since α and C clearly cannot be less than zero, the trends show that, although t_{cc} can be calibrated here, the difference in the minimum central density apparently cannot be completely eliminated.

This analysis should suffice for now. For more precise results in the future, a nested sampling method with > 40 points, for example, would be useful.

5.4.2 Moments Analysis

Other works in the literature [7] suggest that the originally intended fifth moment (i.e. the K_5 approach) is incorrect in the LMFP regime and that alternative moments should be used instead, e.g., Outmezguine’s third moment.

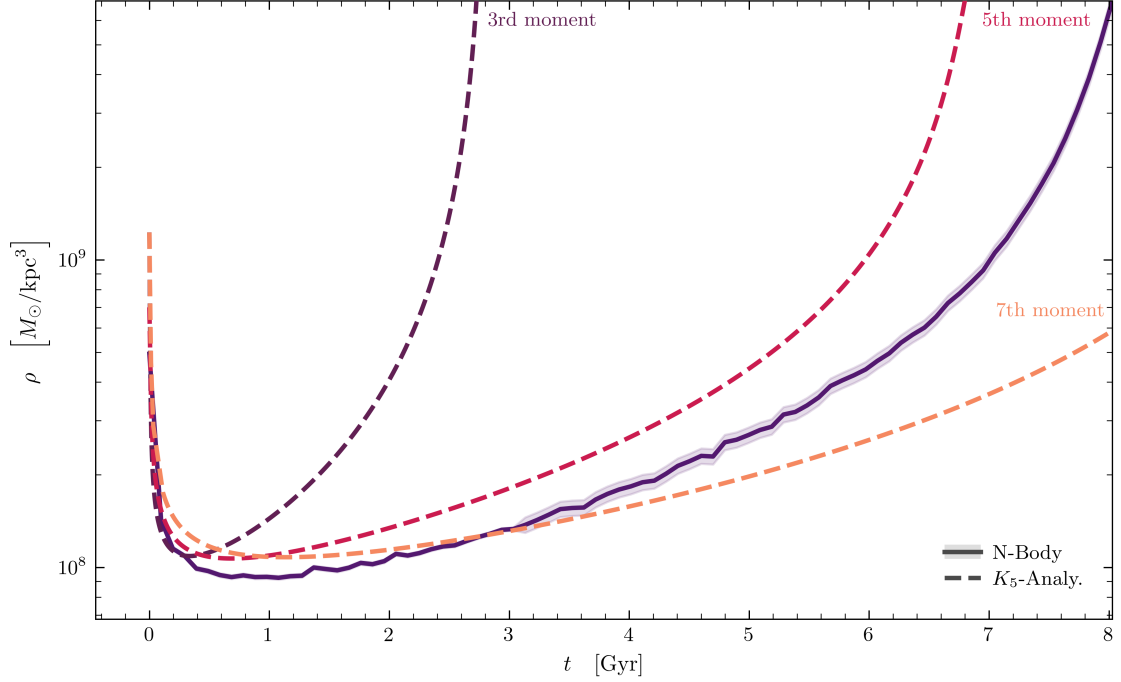


Figure 5.9: Core evolution for the third, fifth, and seventh moments compared to the N -body baseline.

However Figure 5.9, displays the core density development for the third, fifth, and seventh moments. When comparing to Figure 4.1, it is reasonable to assume that the intermediate curves lie within this range.

While adjusting these moments may allow for a better calibration of the core collapse time, it by no means resolves the overall discrepancies. This becomes particularly evident when considering Section 5.1, where the deviation trends for simulations A and B remain relatively similar and only show differences far after core formation. This indicates that no significant systematic error in the LMFP regime is driving these deviations, leaving room for only minor calibrations.

6 Discussion

So far, the theoretical foundations have been laid, methodological preparations have been made, and the resulting findings have been presented. However, a persistent discrepancy between N -body and gravothermal simulations remains that needs further clarification. This chapter addresses precisely this issue.

First, examining the derived basis for the moments K_p examined and followed by an analysis of the implications for the various fluid models. Subsequently, it will be demonstrated why the comparative analysis is rigorous and numerically sound, after which the results will be discussed. In the last section, it will be attempted to identify possible limitations that could account for the results.

6.1 Summary and Interpretation of the Results

This section summarizes the findings and provides interpretations where appropriate; it also draws connections between the findings across sections.

6.1.1 Implications of Velocity-Dependent Cross-Section Mapping

The theoretical velocity moments derived in Sec. 4.1 and illustrated in Fig. 4.1 provided an initial expectation of how the structural evolution of the SIDM-halo is influenced by a velocity-dependent cross-section. Because the central velocity dispersion increases as the halo evolves, the local effective cross-section systematically declines over time in the inner regions. This progressive suppression of scattering events naturally retards energy transport, thereby prolonging the overall core-collapse timescale. When σ_0 is sufficiently large, these evolutionary deviations accumulate over the core-formation epoch, making noticeable discrepancies between the effective model and the dual-fluid model inevitable.

The numerical results presented in Sec. 5.3 fully validate this physical picture while demonstrating exceptional agreement across all tested approaches. Combined with the rigorous initialization protocol established in Sec. 4.2, this consistency provides strong evidence that the velocity-dependent cross-section mapping has been implemented correctly and that the fluid code produces robust, physically reliable solutions in this regime.

6.1.2 Gravothermal Implementation Check

Several chapters of this thesis aim to verify the implementation. First, the mathematical derivation is double-checked and used to reproduce the plots from the

Fischer, Yu, and Dolag [9] paper; next, the fluid code is fine-tuned for convergence and initial conditions; and finally, three different types of implementations are carried out. It is therefore highly certain that the results are correct at the purely computational level.

6.1.3 *N*-body Comparison

Sec. 5.2 performs a direct comparison between *N*-body simulations and gravothermal fluid solutions, highlighting key discrepancies between distinct characteristic timescales (t_{cf} and t_{cc}) and significantly different minimum core densities. It is highly unlikely that mass conservation issues account for these observed deviations. Furthermore, testing various calibrations in Sec. 5.4 failed to resolve or fully explain these differences. Similarly, initiating simulations in different operational regimes, as explored in Sec. 5.1, yielded the same persistent discrepancies.

6.1.4 Conclusion

In summary, while the literature-established formulation for the gravothermal fluid framework was implemented and executed correctly, this work demonstrates that a fundamental discrepancy between *N*-body simulations and fluid models remains unresolved. This leaves three plausible interpretations: either the *N*-body benchmark results suffer from unforeseen systematic errors, which is highly unlikely given their established validation, the gravothermal fluid framework is operating beyond its physical regime of validity, or, most likely, the standard literature approach for mapping phase-space dynamics into fluid models is oversimplified and inadequate.

6.2 Limitations and Solutions

For the sake of completeness, several content limitations must be addressed, including a purely qualitative analysis in 5.3.2, the examination of only a single galaxy setup (ρ_s/r_s), and the restriction to *N*-body data for only two values of σ_0 with no variation in w , despite logical motivation. Furthermore, two additional issues arise concerning the direct comparison, which are presented in detail below and evaluated as potential sources of error.

6.2.1 Anisotropy

The analysis of [9] highlights that even though the *N*-body simulations are initialized isotropically, they naturally develop velocity anisotropy at later evolutionary stages, as measured by the anisotropy parameter $\beta \equiv 1 - \sigma_\theta^2/\sigma_r^2$. The standard gravothermal fluid framework inherently assumes a fully isotropic velocity distribution ($\beta = 0$).

As shown in Fig. 6.1, the anisotropy parameter remains tightly constrained within $\beta \in [-0.1, 0.1]$ across most of the halo, with appreciable deviations occurring only

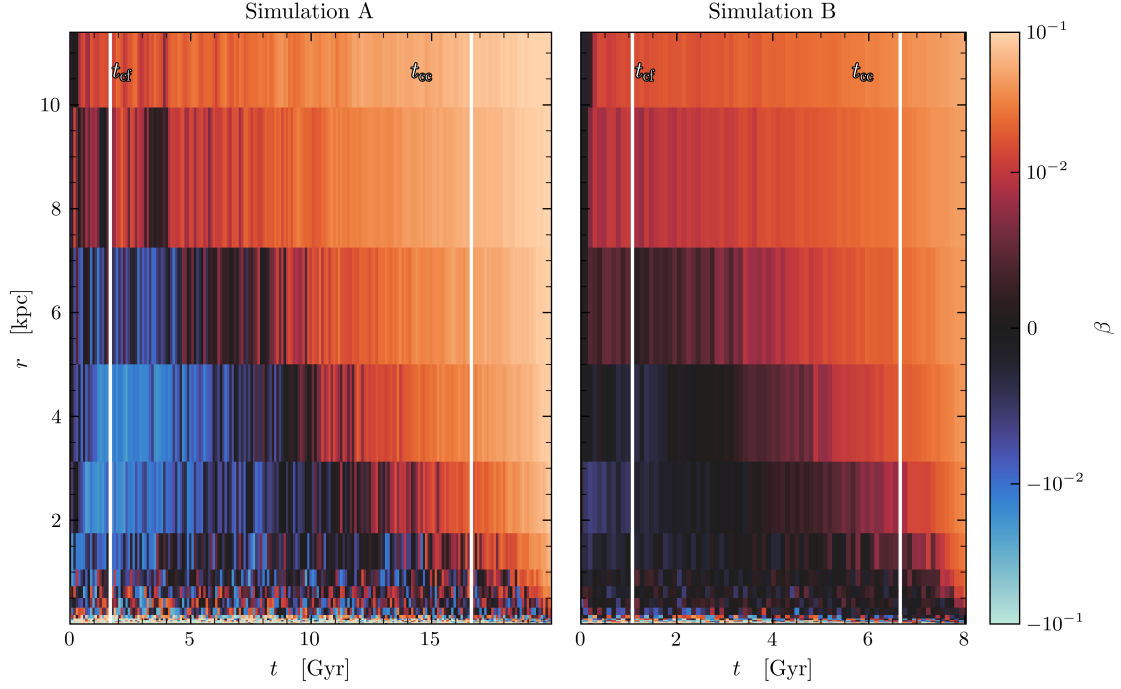


Figure 6.1: Spatial-temporal evolution of the velocity anisotropy parameter $\beta(r, t)$ for Simulation A (left panel) and Simulation B (right panel).

in the outer regions at late times. Crucially, significant discrepancies between the N -body data and fluid approaches particularly in the minimum central density during core formation manifest during early phases when the halo is strictly isotropic. Consequently, incorporating velocity anisotropy into the fluid framework is unlikely to resolve the fundamental discrepancies in the core.

6.2.2 Eddington Initialization

Another key difference lies in the initial velocity distribution: while high-resolution N -body simulations are initialized via Eddington inversion, the gravothermal fluid framework assumes a local Maxwell-Boltzmann distribution.

To determine whether this discrepancy drives deviations in halo evolution, the mathematical foundation must first be established. Following Binney and Tremaine [26], the phase-space distribution function $f(\mathbf{x}, \mathbf{v})$ for a spherically symmetric, isotropic system in dynamical equilibrium depends solely on the relative binding energy per unit mass, $\mathcal{E} \equiv \Psi(r) - \frac{1}{2}v^2$ (where $\Psi(r) \equiv -\Phi(r) + \Phi_0$ is the relative gravitational potential). This function is uniquely determined by the spatial density profile via

$$f(\mathcal{E}) = \frac{1}{\sqrt{8\pi^2}} \left[\int_0^{\mathcal{E}} \frac{d\Psi}{\sqrt{\mathcal{E} - \Psi}} \frac{d^2\rho}{d\Psi^2} + \frac{1}{\sqrt{\mathcal{E}}} \left(\frac{d\rho}{d\Psi} \right)_{\Psi=0} \right]. \quad (6.1)$$

At the initial time step ($t = 0$), the spatial density distribution $\rho(r)$ is given by the analytical NFW profile. The corresponding relative gravitational potential $\Psi(r)$

is fully governed by the enclosed mass distribution, $M(R) = 4\pi \int_0^R \rho(r)r^2 dr$, approaching a Keplerian fall-off ($\Psi \propto 1/r$) in the outer regions.

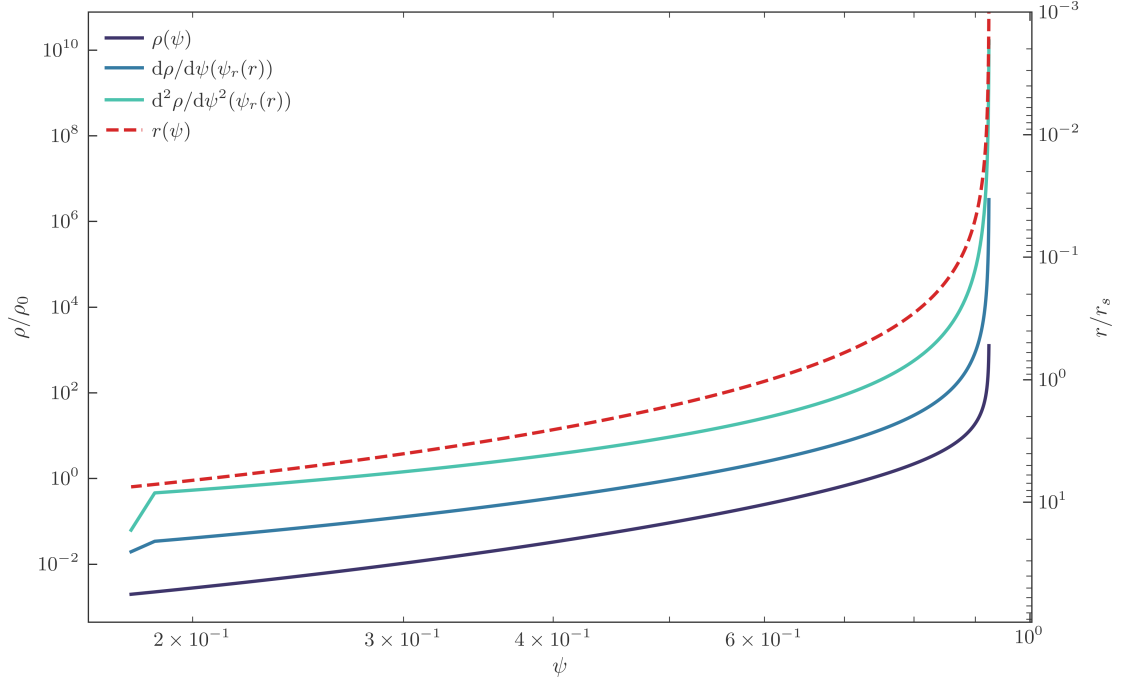


Figure 6.2: Numerical evaluation of the key underlying functions derived via the Eddington inversion for the initial NFW profile.

Fig. 6.2 illustrates the numerical solution of these relations. Notably, the evaluation of $\rho(\Psi)$ exhibits a mild discontinuity at low potentials due to the manual truncation of the outer profile to zero. Although this boundary cutoff introduces minor numerical artifacts in the outer halo, it is acceptable for this initial diagnostic.

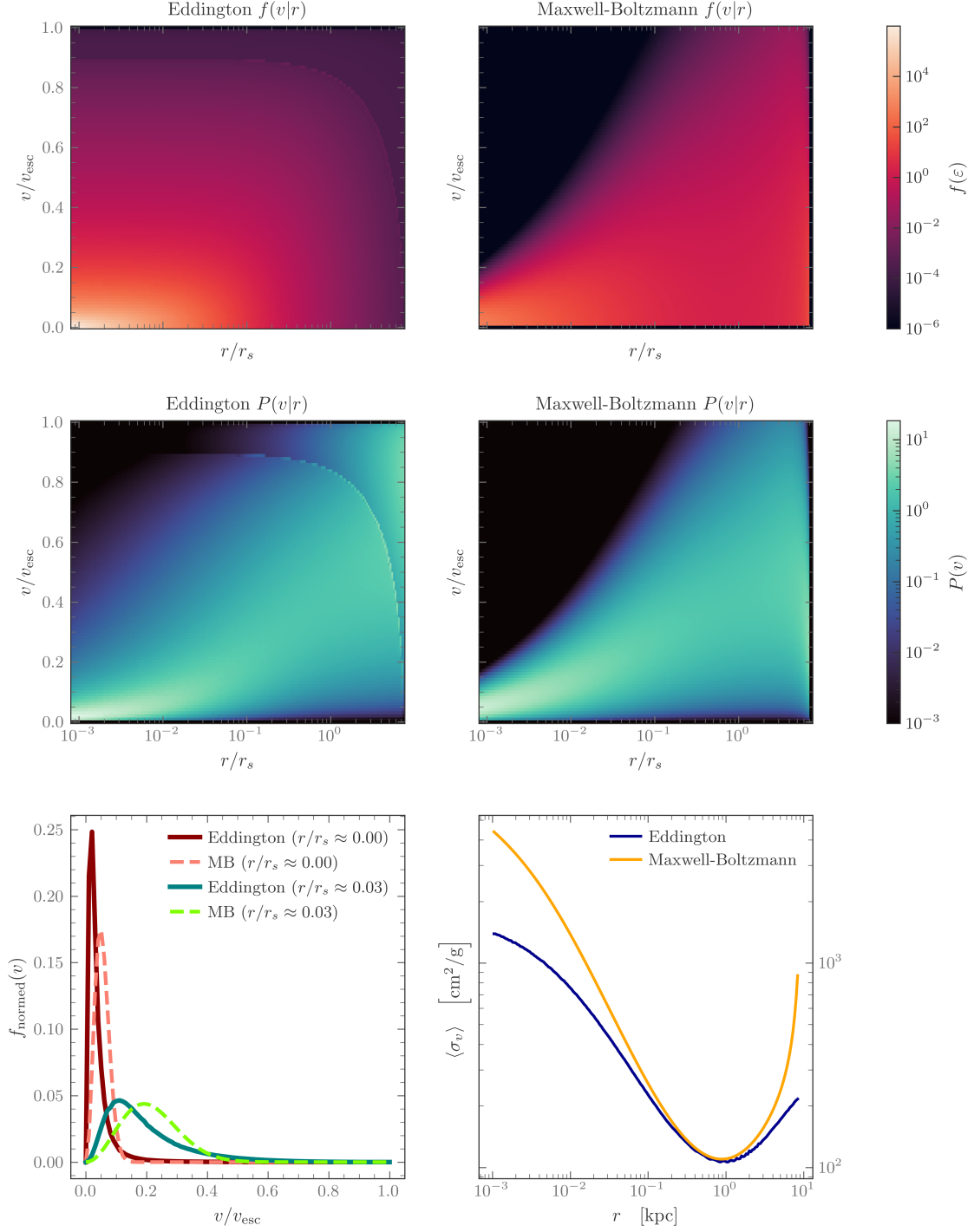


Figure 6.3: Phase space comparison between Eddington inversion and Maxwell-Boltzmann ansatz for the initial NFW profile ($t = 0$). The 2D heatmaps show $f(v|r)$ (top) and $P(v|r)$ (middle) against radius r/r_s and normalized velocity v/v_{esc} . The bottom panels contrast $f_{\text{normed}}(v)$ at selected radii (left) and the resulting radial profiles of $K_p(r)$ (right).

Inserting the derived solutions into Eq. (6.1) and integrating over the two-dimensional grid $r \in (r_{\min}, r_{\max})$ and $v \in (0, v_{\text{escape}})$ yields the explicit phase-space distribution function $f(\mathcal{E})$ at $t = 0$, along with the resulting deviations in the moment factors K_p .

As expected, the Eddington inversion has a clear impact on the inner velocity distribution, systematically lowering the dimensionless factor K_p . Considering that K_p enters directly proportional to the cross-section (recall $\langle \sigma \rangle = \sigma_0 \cdot K_5$), a reduced effective cross-section generally slows the pace of core collapse. Consequently, if this behavior persists over subsequent time steps (and provided the transition to a Maxwell-Boltzmann distribution occurs on a sufficiently long timescale) this effect would inevitably shift t_{cc} to later times.

However, this mechanism alone still fails to account for the discrepancy observed in the minimum core density. Furthermore, adopting a different initial distribution function f fundamentally modifies all governing differential equations from the start. As a result, altered early-time differential equations might induce a different structural evolution of the central density during the initial core-formation phase.

6.3 Future Work

To further elucidate the underlying causes of the observed discrepancies, several natural extensions should be pursued in future work.

First, the purely qualitative analysis in 5.3.2 should be extended to a quantitative one. Second, the examination of only a single galaxy setup (ρ_s, r_s) should be expanded to multiple galaxies, and the restriction to N -body data for only two values of σ_0 with no variation in w should be extended by varying both parameters. Furthermore, another N -body code should be used to validate the results, an alternative fluid code featuring a 2D approach should be employed to investigate anisotropy effects, and solutions should be developed to execute a run using an Eddington-initialized fluid framework.

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Hilfsmittelverzeichnis

Gemäß den Richtlinien zur Eigenständigkeit wissenschaftlicher Arbeiten werden nachfolgend alle eingesetzten KI-gestützten Werkzeuge sowie Übersetzungstools detailliert mit ihrem jeweiligen Zweck und dem Ausmaß ihrer Nutzung offengelegt.

Tool / Software	Sections	Purpose / Type of Use	Concrete Procedure & Extent
Google Gemini	Chapters 1 to 7	Proofreading & Reviewer Feedback	Submitted original drafts to check grammar, style, and argumentative logic. All text was written by me; suggestions were reviewed and incorporated. Final content and linguistic reviews were conducted by human readers.
Google Gemini	Chapters 1 to 7	Literature Search	Used to identify relevant academic literature and references. No text content was taken from Gemini; all sources were read independently and cited directly from the original works.
Google Gemini	See GitHub Repository	Code Debugging / Troubleshooting	Inputted original code snippets alongside error messages for root-cause analysis and syntax fixes. The algorithm and overall architecture were designed independently.
DeepL / DeepL Write	Chapters 1 to 7	Translation & Linguistic Fine-Tuning	Selected translations of technical terms/sentences and final stylistic corrections of my own sentence structures.

Table 8.1: Overview of AI and AI-Assisted Tools Used

Ehrenwörtliche Erklärung / Statutory Declaration

Erklärung nach § 39 (15) Prüfungsordnung 2020 für den BAMA-Studiengang Physik:

Hiermit erkläre ich, dass ich die Arbeit selbstständig und ohne Benutzung anderer als der angegebenen Quellen und Hilfsmittel verfasst habe. Alle Stellen der Arbeit, die wörtlich oder sinngemäß aus Veröffentlichungen oder aus anderen fremden Texten entnommen wurden, sind von mir als solche kenntlich gemacht worden. Ferner erkläre ich, dass die Arbeit nicht – auch nicht auszugsweise – für eine andere Prüfung verwendet wurde.

Frankfurt am Main, 22. September 2026

Unterschrift (Lukas Lehmann)